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Two-time scales in spatially structured models of population dynamics: A semigroup approach

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The aim of this work is to provide a unified approach to the treatment of a class of spatially structured population dynamics models whose evolution processes occur at two different time scales. In the setting of the C_0-semigroup theory, we will consider a general formulation of some semilinear evolution problems defined on a Banach space in which the two-time scales are represented by a parameter $\varepsilon > 0$ small enough, that mathematically gives rise to a singular perturbation problem. Applying the so-called *aggregation of variables* method, a simplified model called the *aggregated model* is constructed. A nontrivial mathematical task consists of comparing the asymptotic behaviour of solutions to both problems when $\varepsilon \to 0_+$, under the assumption that the aggregated model has a compact attractor. Applications of the method to a class of two-time reaction–diffusion models of spatially structured population dynamics and to models with discrete spatial structure are given.

1. Introduction

The aim of this paper is to provide a unified approach to the treatment of a class of models of spatially structured population dynamics whose evolution processes occur at two different time scales: a slow one for the demography and a fast one for the migrations. In the setting of the C_0-semigroup theory, we will consider a general formulation of some semilinear evolution problems in which the two-time scales are represented by a parameter $\varepsilon > 0$ small enough that mathematically gives rise to a singular perturbation problem and to which the so-called *aggregation of variables method* can be applied. This method allows the construction of a simplified model called the *aggregated model*, the proof of comparison results between the behaviour of solutions to both problems being a nontrivial mathematical task, so that conclusions on the initial complex model could be deduced from an analysis of the aggregated one.

In this work we provide a method to construct the aggregated model of an abstract two-time semilinear evolution equation defined on a Banach space, establishing a general comparison result in the case where the simplified model has a local compact attractor. We illustrate the method with some applications to two-time reaction–diffusion models of spatially structured populations.

In recent decades there has been a lot of interest in spatial dynamics of ecological systems (see [10,26,32], among others), giving rise to several ways to introduce space in mathematical models of population dynamics. A first approach consists of considering a discrete space: the environment is considered as a set of discrete patches connected by migration and the

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evolution processes are described by a set of ordinary differential equations (O.D.E.) taking into account local interactions on each patch (birth, death, trophic interactions) and migration terms describing patch changes. Most models consider the simplest case of density-independent migrations (see [1,19,29]) and, to our knowledge, still, few works have addressed the effects of density-dependent dispersal. We could, nevertheless, cite the following ones: [3,2,9,20]. In a more recent work [21], some of us considered a predator-prey model with two patches connected by density-dependent migrations. Assuming that prey would be more likely to leave a patch when the local patch predator density is large and as well predators would rather stay on a patch when the local prey density is large, the model shows that density-dependent migrations have important effects on the dynamics and the stability of the prey-predator community.

This discrete patch approach is limited by the number of patches that can be considered, since it is linked with the number of equations of the model; indeed, most models consider typically two patches connected by migrations. However, in more realistic cases, the environment cannot be limited to a system of few patches. In order to be able to take into account a big number of discrete patches, several authors introduced two-time scales in the model: a fast time scale corresponding to patch changes and a slow one associated with local interactions in each patch [30]. Aggregation methods can take advantage of these different time scales to reduce the complexity of an initial model formulated as a large system of nonlinear O.D.E., allowing the construction of a reduced model governed by a few global variables, keeping the individual features of the dynamics, and mathematically is more tractable [25,24]. To our knowledge, perfect and approximate aggregation methods were introduced for the first time in population dynamics in [17] and [18] and we refer the reader to [5] and [6] as good and recent reviews of these methods and their applications to different aspects of theoretical ecology.

Another way to introduce the spatial structure in mathematical modelling consists of considering a continuous space, which usually leads to reaction–diffusion models, formulated as a set of partial differential equations (P.D.E.). The aim of this paper is to extend aggregation methods to this setting. In particular, we will consider continuously spatially distributed populations in which the diffusion takes place at a faster time scale than local growth. Choosing the total population as a new variable, we can reduce the model to an O.D.E. governing the dynamics without losing the individual features.

The organisation of the paper is as follows: Section 2 contains the main results of the paper. First, a general method to construct an aggregated model that simplifies a two-time semilinear evolution equation defined on a Banach lattice is explained. Secondly, a comparison result between the solutions to both problems is established: the existence of a compact attractor for the aggregated model assures the existence of a compact attractor for the perturbed model, that approaches the aggregated one for $\varepsilon > 0$ small enough. Section 3 applies the general theory to reaction–diffusion models with two-time scales, recovering in some particular cases well-known results on parabolic reaction–diffusion equations with large diffusivity. Section 4 is dedicated to make evident that the abstract formulation covers also simpler situations like models with discrete spatial structure. The work ends with two sections of conclusions and references.

2. Aggregation of variables in a two-time semilinear evolution differential equation

Our main goal in this section is the application of the aggregation of variables method to a two-time semilinear evolution equation defined on a Banach lattice for which the Perron–Frobenius theory on positive C_0-semigroups holds. The reader can find the main theoretical results that we will apply in Refs. [4,23,27,34], among others.

To be precise, let us consider the following Cauchy problem for an abstract semilinear parabolic differential equation defined on a Banach lattice $(X, \|\cdot\|_X)$:

$$(\mathrm{CP})_\varepsilon \quad \begin{cases} n'_\varepsilon(t) = \dfrac{1}{\varepsilon} A n_\varepsilon(t) + \mathcal{F}\big(n_\varepsilon(t)\big), & t > 0, \\ n_\varepsilon(0) = n_0 \end{cases}$$

where $\varepsilon > 0$ is a small parameter and we assume that operators A and $\mathcal{F}$ satisfy the following hypotheses:

Hypothesis 1. The operator $A : D(A) \subset X \longrightarrow X$ is the infinitesimal generator of a C_0-semigroup $\{T_0(t)\}_{t \geqslant 0}$ defined on X.

Hypothesis 2. The nonlinear operator $\mathcal{F} : X \longrightarrow X$ is locally Lipschitz continuous. That is, for each $\gamma > 0$ there exists a constant $L_\gamma > 0$ such that for each $\varphi_i \in X$ with $\|\varphi_i\|_X \leqslant \gamma$, $i = 1, 2$, the following holds:

$$\big\|\mathcal{F}(\varphi_1) - \mathcal{F}(\varphi_2)\big\|_X \leqslant L_\gamma \|\varphi_1 - \varphi_2\|_X.$$

With the help of the variation of constants formula, the differential problem $(\mathrm{CP})_\varepsilon$ can be transformed into the integral equation

$$n_\varepsilon(t) = T_\varepsilon(t) n_0 + \int_0^t T_\varepsilon(t-\sigma)\big(\mathcal{F}\big(n_\varepsilon(\sigma)\big)\big)\, d\sigma, \quad t \geqslant 0 \tag{1}$$

where we have introduced the *rescaled* semigroup $T_\varepsilon(t) := T_0((1/\varepsilon)t)$, which takes into account the factor $1/\varepsilon$ of the model.

As usual, the notation $C([0, T]; X)$ $(T > 0)$ represents the Banach space of continuous functions $n : [0, T] \longrightarrow X$, endowed with the norm $\|n\|_C := \sup_{t \in [0,T]} \|n(t)\|_X$. Then, a *classical* solution to $(\mathrm{CP})_\varepsilon$ is a function $n_\varepsilon \in C([0, T]; X)$ for some $T > 0$

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such that n_ε is continuously differentiable on $(0,T)$, $n_\varepsilon(t) \in D(A)$ for $t > 0$, and satisfies $(\mathrm{CP})_\varepsilon$. A function $n_\varepsilon \in C([0,T];X)$ which satisfies (1) for $t \in [0,T]$ is called a *mild solution* to $(\mathrm{CP})_\varepsilon$.

The standard theory on abstract semilinear parabolic differential equations can be applied, yielding the following result, which is an immediate consequence of Theorems 6.1.4 and 6.1.5 [27]:

Theorem 1. *Under Hypotheses* 1 *and* 2, *for each initial data $n_0 \in X$ there exists a unique n_ε mild solution to* $(\mathrm{CP})_\varepsilon$ *defined on a maximal interval* $[0, T_{\max})$, $T_{\max} > 0$. *Moreover, if $T_{\max} < +\infty$, then* $\lim_{t \to T_{\max -}} \|n_\varepsilon(t)\|_X = +\infty$.

If $\mathcal{F}$ is continuously Fréchet-differentiable and the initial data $n_0 \in D(A)$, then n_ε is the classical solution to $(\mathrm{CP})_\varepsilon$.

2.1. Construction of an aggregated model

With the help of the theory of positive C_0-semigroups we will proceed to reduce the abstract evolution equation to an ordinary differential equation whose solutions serve to approximate the initial complex dynamics, for $\varepsilon > 0$ small enough. As we are interested in covering population dynamics models that involve several populations, previously we establish a vector formulation of the so-called *perturbed problem* $(\mathrm{CP})_\varepsilon$. That is, in all this paper we will assume that $X := E^q$, $q \geqslant 1$, where $(E, \|\cdot\|_E)$ is a Banach lattice and X is endowed with the product norm. Moreover we also assume that $A := \mathrm{diag}(A_1, \ldots, A_q)$ where $A_j : D(A_j) \subset E \longrightarrow E$ is the infinitesimal generator of a C_0-semigroup $\{T_{0j}(t)\}_{t \geqslant 0}$ defined on E, $j = 1, \ldots, q$, so that $T_0(t) := \mathrm{diag}(T_{01}(t), \ldots, T_{0q}(t))$. Then, we impose the following condition, that will become essential in the subsequent development:

Hypothesis 3. For each $j = 1, \ldots, q$, the semigroup $\{T_{0j}(t)\}_{t \geqslant 0}$ is eventually compact, positive and irreducible.

Moreover, the spectral bound of A_j, $s(A_j) := \sup\{\mathrm{Re}\,\lambda,\ \lambda \in \sigma(A_j)\}$, satisfies that $s(A_j) = 0$. As usual, $\sigma(A_j)$ stands for the spectrum of operator A_j.

The main results concerning our work are summarised in the following theorem (see [4,23]):

Theorem 2. *Under Hypotheses* 1 *and* 3, *for each $j = 1, \ldots, q$ the following hold*:

i) *$s(A_j) = 0$ is an isolated point of $\sigma(A_j)$ which is strictly dominant in real part. Therefore, there exists $\alpha_j^* > 0$ such that*

$$\sigma(A_j) = \{0\} \cup \Lambda_j; \quad \Lambda_j \subset \left\{z \in \mathbf{C};\ \mathrm{Re}\, z < -\alpha_j^*\right\}.$$

ii) $\dim\ker A_j = 1$ *and there exist $\mu_j > 0$, $\mu_j \in \ker A_j$ and a strictly positive functional $\mu_j^* \in \ker A_j^*$ such that $\langle \mu_j^*, \mu_j \rangle = 1$, where A_j^* is the adjoint operator of A_j and $\langle \cdot,\cdot \rangle$ stands for the duality (E^*, E).*

iii) *There exists a direct sum decomposition*

$$E = \ker A_j \oplus S_j; \quad S_j := \mathrm{Im}\, A_j \tag{2}$$

which reduces A_j and the semigroup $\{T_{0j}(t)\}_{t \geqslant 0}$. That is, $\ker A_j$ *and S_j are closed invariant subspaces under A_j and $T_{0j}(t)$, $t \geqslant 0$. Moreover, $\sigma(A_{jS}) = \Lambda_j$ and $\|T_{jS}(t)\| \leqslant M_S e^{-\alpha_j^* t}$, $t > 0$, where A_{jS} and $T_{jS}(t)$ represent respectively the restriction of A_j and $T_{0j}(t)$ to S_j.*

iv) *In the direct sum decomposition* (2) *we have*

$$\mathrm{Im}\, A_j = \left\{\varphi \in E;\ \left\langle \mu_j^*, \varphi \right\rangle = 0\right\}$$

and the associated projection onto $\ker A_j$ *is given by*

$$\forall \psi \in E, \quad \Pi_{Aj}\psi := \left\langle \mu_j^*, \psi \right\rangle \mu_j.$$

In turn, these results can be adapted to the product space X, yielding the following:

a) Set $\alpha^* := \min(\alpha_1^*, \ldots, \alpha_q^*) > 0$. Then,

$$\sigma(A) = \{0\} \cup \Lambda; \quad \Lambda \subset \left\{z \in \mathbf{C};\ \mathrm{Re}\, z < -\alpha^*\right\}.$$

b) $\dim\ker A = q$ and $\ker A$ is spanned by the set $\{(\mu_1, 0, \ldots, 0)^T, \ldots, (0, 0, \ldots, \mu_q)^T\}$.

c) The direct sum decomposition

$$X = \ker A \oplus S; \quad S := \mathrm{Im}\, A \tag{3}$$

reduces A and the semigroup $\{T_0(t)\}_{t \geqslant 0}$. Moreover, $\sigma(A_S) = \Lambda$ and $\|T_S(t)\| \leqslant M_S e^{-\alpha^* t}$, $t > 0$, where A_S and $T_S(t)$ represent respectively the restriction of A and $T_0(t)$ to S.

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d) In the direct sum decomposition (3) we have

$$\operatorname{Im} A = \{\varphi := (\varphi_1, \ldots, \varphi_q)^T \in X;\ \langle \mu_j^*, \varphi_j \rangle = 0,\ j = 1, \ldots, q\}$$

and the associated projection onto $\ker A$ is given by

$$\forall \psi := (\psi_1, \ldots, \psi_q)^T \in X, \quad \Pi_A \psi := \left(\langle \mu_1^*, \psi_1 \rangle \mu_1, \ldots, \langle \mu_q^*, \psi_q \rangle \mu_q \right)^T.$$

Notice that $(\mathrm{CP})_\varepsilon$ is in fact a system of q semilinear evolution equations. To be precise, since $n_\varepsilon(t) := (n_\varepsilon^1(t), \ldots, n_\varepsilon^q(t))^T$ and also $\mathcal{F}(\varphi) := (\mathcal{F}_1(\varphi), \ldots, \mathcal{F}_q(\varphi))^T$ where each $\mathcal{F}_j : X \longrightarrow E$ is a locally continuous Lipschitz operator, the perturbed problem can be written as

$$\left(n_\varepsilon^j\right)'(t) = \frac{1}{\varepsilon} A_j n_\varepsilon^j(t) + \mathcal{F}_j\left(n_\varepsilon(t)\right), \quad j = 1, \ldots, q.$$

The underlying idea in the construction of an aggregated model consists of projecting the dynamics of $(\mathrm{CP})_\varepsilon$ onto $\ker A$. To this end, we choose q new variables called *global variables* defined by

$$N_\varepsilon^j(t) := \langle \mu_j^*, n_\varepsilon^j(t) \rangle, \quad j = 1, \ldots, q; \qquad N_\varepsilon(t) := \left(N_\varepsilon^1(t), \ldots, N_\varepsilon^q(t)\right)^T \in \mathbf{R}^q$$

which satisfy

$$\left(N_\varepsilon^j\right)'(t) = \langle \mu_j^*, \left(n_\varepsilon^j\right)'(t) \rangle = \left\langle \mu_j^*, \frac{1}{\varepsilon} A_j n_\varepsilon^j(t) \right\rangle + \langle \mu_j^*, \mathcal{F}_j\left(n_\varepsilon(t)\right) \rangle$$
$$= \langle \mu_j^*, \mathcal{F}_j\left(n_\varepsilon(t)\right) \rangle, \quad j = 1, \ldots, q.$$

Notice that the right-hand side of these equations depends on $n_\varepsilon(t)$. To avoid this difficulty, we substitute it by its projection onto $\ker A$:

$$n_\varepsilon^j(t) \approx N^j(t) \mu_j, \quad j = 1, \ldots, q$$

so that we approximate the initial perturbed model, which is a functional differential equation defined on a Banach lattice, by the *aggregated model* which is a system of coupled nonlinear ordinary differential equations (O.D.E.) in $\mathbf{R}^q$:

$$\left(N^j\right)'(t) = \langle \mu_j^*, \mathcal{F}_j\left(\left(N^1(t)\mu_1, \ldots, N^q(t)\mu_q\right)^T\right) \rangle, \quad j = 1, \ldots, q$$

completed with the initial value:

$$N^j(0) = \langle \mu_j^*, n_\varepsilon^j(0) \rangle = \langle \mu_j^*, n_0^j \rangle, \quad j = 1, \ldots, q.$$

Hypothesis 2 assures that the general theory on solutions to O.D.E. applies to the aggregated model.

To simplify, let us introduce the notations:

$$N(t)\mu := \left(N^1(t)\mu_1, \ldots, N^q(t)\mu_q\right)^T; \qquad \langle \mu^*, \varphi \rangle := \left(\langle \mu_1^*, \varphi_1 \rangle, \ldots, \langle \mu_q^*, \varphi_q \rangle\right)^T.$$

Then, the aggregated model can be written as

$$N'(t) = \langle \mu^*, \mathcal{F}\left(N(t)\mu\right) \rangle; \qquad N(0) = \langle \mu^*, n_0 \rangle. \tag{4}$$

2.2. Comparison result between the asymptotic behaviour of solutions to $(\mathrm{CP})_\varepsilon$ *and the aggregated model*

The goal of this section is to derive comparison results between the solutions to the perturbed and the aggregated models that allow us to conclude that, for $\varepsilon \to 0_+$, the asymptotic behaviour of solutions to $(\mathrm{CP})_\varepsilon$ can be approximated by the behaviour of solutions to the aggregated model. To be precise, assuming that the aggregated model has a compact attractor, we will show that the perturbed model has, for $\varepsilon > 0$ small enough, a compact attractor which is *close* to the aggregated one. This result needs some a priori estimations on the solutions to both problems that we will prove under suitable additional smoothness conditions for the semigroup $\{T_0(t)\}_{t \geqslant 0}$.

Recall (see [12]) that a set $\mathcal{A} \subset \mathbf{R}^q$ is a compact attractor for the aggregated model if it is an invariant compact set and there exists a neighbourhood U of $\mathcal{A}$ such that the ω-limit set of U is $\mathcal{A}$.

Roughly speaking, we start by showing that for $\varepsilon > 0$ small enough, the solutions to $(\mathrm{CP})_\varepsilon$ which start *close* to $\mathcal{A}\mu := \{N\mu;\ N \in \mathcal{A}\}$, remain close to it for all $t \geqslant 0$. To this end, we project the perturbed equation onto $\ker A$ and S.

According to the decomposition given by (3), we can write the solution to $(\mathrm{CP})_\varepsilon$ as

$$n_\varepsilon(t) = N_\varepsilon(t)\mu + \rho_\varepsilon(t), \quad t > 0$$

where $\langle \mu^*, \rho_\varepsilon(t)\rangle = 0$, $\forall t > 0$. This implies that $\langle \mu^*, \rho'_\varepsilon(t)\rangle = 0$, $\forall t > 0$, which in turn yields the following decomposition of $(\mathrm{CP})_\varepsilon$:

$$\begin{cases} N'_\varepsilon(t) = \langle \mu^*, \mathcal{F}(N_\varepsilon(t)\mu + \rho_\varepsilon(t))\rangle, \\ \rho'_\varepsilon(t) = (1/\varepsilon)A_S\rho_\varepsilon(t) + \mathcal{F}_S(N_\varepsilon(t)\mu + \rho_\varepsilon(t)) \end{cases}$$

where A_S, $\mathcal{F}_S$ stand for the projection of operators A and $\mathcal{F}$ on S, respectively.

Decomposing the initial data as $n_0 = N_0\mu + \rho_0$, and using the variation of constants formula, we can write the mild version of the second equation in the above system:

$$\rho_\varepsilon(t) = T_{\varepsilon S}(t)\rho_0 + \int_0^t T_{\varepsilon S}(t-\sigma)\mathcal{F}_S(N_\varepsilon(\sigma)\mu + \rho_\varepsilon(\sigma))\,d\sigma \tag{5}$$

where $T_{\varepsilon S}(t) := T_S((1/\varepsilon)t)$.

For each $W(\mathcal{A})$ neighbourhood of $\mathcal{A}$ in $\mathbf{R}^q$ and $\delta > 0$, we use the following notation to represent a neighbourhood of $\mathcal{A}\mu$ in X:

$$\mathcal{N}(W(\mathcal{A});\delta) := \{N\mu + \rho;\ N \in W(\mathcal{A}),\ \rho \in S,\ \|\rho\|_X < \delta\}.$$

The following proposition contains a result that will be relevant for our purposes:

Proposition 1. *Under Hypotheses* 1, 2 *and* 3, *assume that there exists a compact attractor $\mathcal{A}$ for the aggregated model* (4). *Then, fixing any neighbourhood $W(\mathcal{A})$ and $\delta > 0$, there exist a neighbourhood $W^*(\mathcal{A}) \subset W(\mathcal{A})$, $\delta^* \in (0,\delta)$ and $\varepsilon^* > 0$ such that for all $\varepsilon \in (0,\varepsilon^*)$ and $n_0 = N_0\mu + \rho_0 \in \mathcal{N}(W^*(\mathcal{A});\delta^*)$, the solution to* $(\mathrm{CP})_\varepsilon$ *$n_\varepsilon(t) := N_\varepsilon(t)\mu + \rho_\varepsilon(t)$ such that $n_\varepsilon(0) = n_0$ is defined for all $t \geqslant 0$ and satisfies the following*:

i) $n_\varepsilon(t) \in \mathcal{N}(W^*(\mathcal{A});\delta)$;
ii) $\|\rho_\varepsilon(t)\|_X \leqslant C_1 e^{-\beta t/\varepsilon}\|\rho_0\|_E + C_2\varepsilon$

for some positive constants $C_1, C_2 > 0$ (nondependent on $W(\mathcal{A})$, δ) and any $\beta \in (0,\alpha^*)$, where $\alpha^* > 0$ is the constant mentioned in Theorem 2.

Proof. First of all, notice that (see [11]) there exist a bounded neighbourhood of $\mathcal{A}$, $U_0(\mathcal{A}) \subset \mathbf{R}^q$ and a continuous Lipschitz scalar function $V : U_0(\mathcal{A}) \longrightarrow \mathbf{R}$ such that $\forall N \in U_0(\mathcal{A})$ the following hold:

i) $V(N^*) = 0$, $\forall N^* \in \mathcal{A}$;
ii) there exist two real-valued nonnegative and continuous functions $a(r)$, $b(r)$, with $a(r) > 0$ if $r > 0$, $b(0) = 0$, $a(r)$ nondecreasing such that $a(\mathrm{dist}(N,\mathcal{A})) \leqslant V(N) \leqslant b(\mathrm{dist}(N,\mathcal{A}))$;
iii) $\dot V_{(4)}(N) \leqslant -V(N)$

where $\dot V_{(4)}(N)$ represents the *derivative of V along the solutions to* (4). That is

$$\dot V_{(4)}(N_0) := \limsup_{h\to 0_+} \frac{V(N(h;N_0)) - V(N_0)}{h}$$

with $N(t;N_0)$ being the solution to (4) such that $N(0;N_0) = N_0$.

Let $W(\mathcal{A})$, $\delta > 0$, be fixed. Without loss of generality, we can assume that $W(\mathcal{A})$ is open and bounded with $\overline{W(\mathcal{A})} \subset U_0(\mathcal{A})$ and also that $0 < \delta < \delta_0$, for some $\delta_0 > 0$ fixed. Condition (ii) assures that $k_0 := \min\{V(N),\ N \in \partial W(\mathcal{A})\}$ satisfies that $k_0 > 0$, where $\partial W(\mathcal{A})$ represents the boundary of $W(\mathcal{A})$, which is a compact set in $\mathbf{R}^q$. Choosing $\eta \in (0,k_0)$, we define $W^*(\mathcal{A}) := \{N \in W(\mathcal{A});\ V(N) < \eta\}$, which is an open neighbourhood of $\mathcal{A}$ such that $\mathcal{A} \subset W^*(\mathcal{A}) \subset W(\mathcal{A})$.

Then, let us consider the solution to $(\mathrm{CP})_\varepsilon$ $n_\varepsilon(t) = N_\varepsilon(t)\mu + \rho_\varepsilon(t)$ corresponding to an initial data $n_0 = N_0\mu + \rho_0 \in \mathcal{N}(W^*(\mathcal{A});\delta^*)$, for any $\delta^* \in (0,\delta)$, which is defined on some maximal interval $[0, T_{\max})$. By continuity of solutions, there exists $t > 0$ such that for all $s \in [0,t]$, we have $n_\varepsilon(s) \in \mathcal{N}(W^*(\mathcal{A});\delta^*)$.

To simplify the notation we will denote by C_i, $i = 1,2,\ldots$ the positive constants that appear in the calculations and whose specific values are not relevant. Since the solution $n_\varepsilon(s)$, $s \in [0,t]$, belongs to the fixed bounded set $\mathcal{N}(U_0(\mathcal{A});\delta_0)$, Hypothesis 2 together with the compactness of $\mathcal{A}$ provide the following estimation, independent of $W(\mathcal{A})$, δ and ε:

$$\begin{aligned} \|\mathcal{F}(n_\varepsilon(s))\|_X &\leqslant \sup_{N^*\in\mathcal{A}} \|\mathcal{F}(N_\varepsilon(s)\mu + \rho_\varepsilon(s)) - \mathcal{F}(N^*\mu)\|_X + \sup_{N^*\in\mathcal{A}} \|\mathcal{F}(N^*\mu)\|_X \\ &\leqslant C_1 \sup_{N^*\in\mathcal{A}} |N_\varepsilon(s) - N^*| + C_2\|\rho_\varepsilon(s)\|_X + \sup_{N^*\in\mathcal{A}} \|\mathcal{F}(N^*\mu)\|_X \\ &\leqslant C_1 + C_2\|\rho_\varepsilon(s)\|_X. \end{aligned} \tag{6}$$

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Then, formula (5) together with Theorem 2(iii) yield:

$$\begin{aligned}\|\rho_\varepsilon(s)\|_X &\leqslant M_S e^{-\alpha^* s/\varepsilon}\|\rho_0\|_X + M_S\int_0^s e^{-\alpha^*(s-\sigma)/\varepsilon}\big\|\mathcal{F}_S\big(N_\varepsilon(\sigma)\mu+\rho_\varepsilon(\sigma)\big)\big\|_X\,d\sigma\\ &\leqslant M_S e^{-\alpha^* s/\varepsilon}\|\rho_0\|_X + C_1\varepsilon + C_2\int_0^s e^{-\alpha^*(s-\sigma)/\varepsilon}\big\|\rho_\varepsilon(\sigma)\big\|_X\,d\sigma.\end{aligned}$$

Let $\beta \in (0,\alpha^*)$ be a fixed constant and set

$$v_\varepsilon(s) := \big\|\rho_\varepsilon(s)\big\|_X e^{\beta s/\varepsilon}, \qquad W_\varepsilon(s) := \sup_{\tau\in[0,s]} v_\varepsilon(\tau).$$

Then, we have

$$v_\varepsilon(s) \leqslant M_S e^{-(\alpha^*-\beta)s/\varepsilon}\|\rho_0\|_X + C_1\varepsilon e^{\beta s/\varepsilon} + C_2\varepsilon W_\varepsilon(s)$$

and therefore

$$W_\varepsilon(s) = \sup_{\tau\in[0,s]} v_\varepsilon(\tau) \leqslant M_S\|\rho_0\|_X + C_1\varepsilon e^{\beta s/\varepsilon} + C_2\varepsilon W_\varepsilon(s).$$

Choosing ε_0 so that $C_2\varepsilon_0 < 1$, we have $\forall\varepsilon\in(0,\varepsilon_0)$:

$$W_\varepsilon(s) \leqslant \frac{M_S}{1-C_2\varepsilon}\|\rho_0\|_X + \frac{C_1\varepsilon}{1-C_2\varepsilon}e^{\beta s/\varepsilon} \leqslant C_3\|\rho_0\|_X + C_4\varepsilon e^{\beta s/\varepsilon}$$

which finally provides the estimation:

$$\big\|\rho_\varepsilon(s)\big\|_X \leqslant C_1\|\rho_0\|_X e^{-\beta s/\varepsilon} + C_2\varepsilon, \quad s\in[0,t],\ 0<\varepsilon<\varepsilon_0. \tag{7}$$

Now we proceed to estimate the derivative of V along the solution $N_\varepsilon(s)$, that is

$$\begin{aligned}\dot V_{(\varepsilon)}\big(N_\varepsilon(s)\big) &:= \limsup_{h\to 0_+}\frac{V(N_\varepsilon(h;N_\varepsilon(s))) - V(N_\varepsilon(s))}{h}\\ &\leqslant \limsup_{h\to 0_+}\frac{V(N(h;N_\varepsilon(s))) - V(N_\varepsilon(s))}{h} + \limsup_{h\to 0_+}\frac{V(N_\varepsilon(h;N_\varepsilon(s))) - V(N(h;N_\varepsilon(s)))}{h}\\ &\leqslant \dot V\big(N_\varepsilon(s)\big) + C_1\big\|\mathcal{F}\big(N_\varepsilon(s)\mu+\rho_\varepsilon(s)\big) - \mathcal{F}\big(N_\varepsilon(s)\mu\big)\big\|_X\\ &\leqslant \dot V\big(N_\varepsilon(s)\big) + C_1\big\|\rho_\varepsilon(s)\big\|_X\\ &\leqslant -V\big(N_\varepsilon(s)\big) + C_1\|\rho_0\|_X + C_2\varepsilon\end{aligned} \tag{8}$$

which leads to the following inequality, valid $\forall s\in[0,t]$ and $\forall\varepsilon\in(0,\varepsilon_0)$:

$$V\big(N_\varepsilon(s)\big) \leqslant e^{-s}V(N_0) + \big(C_1\|\rho_0\|_X + C_2\varepsilon\big)\big(1-e^{-s}\big). \tag{9}$$

Choosing $\delta^*>0$, $\varepsilon^*\in(0,\varepsilon_0)$ small enough so that $C_1\delta^* + C_2\varepsilon^* < \min(\eta,\delta)$, define

$$t^* := \sup\big\{t;\ N_\varepsilon(s)\in W^*(\mathcal{A}),\ \forall s\in[0,t]\big\}$$

being evident that $t^*\in(0,T_{\max}]$. Assume that $t^*<+\infty$. If $t^*=T_{\max}$, then $\lim_{t\to t^*_-}\|n_\varepsilon(t)\|_X=+\infty$, which is not possible since $n_\varepsilon(t)$ belongs to a bounded set. If $t^*<T_{\max}$, then $n_\varepsilon(t^*)$ is defined and (9) implies that $V(N_\varepsilon(t^*))<\eta$, which contradicts the definition of t^*. Therefore, $t^*=+\infty$, which in turn implies that $n_\varepsilon(t)$ is defined for all $t\geqslant 0$ and that $N_\varepsilon(t)\in W^*(\mathcal{A})$, together with $\|\rho_\varepsilon(t)\|_X<\delta$, for all $t\geqslant 0$, as we wanted to prove. □

The next step consists of proving that there exists a set of initial conditions whose omega-limit set is a compact attractor for $(\mathrm{CP})_\varepsilon$. To this end we need to assure the precompactness of the corresponding positive orbits, which we will do imposing supplementary smoothness conditions to the semigroup $\{T_0(t)\}_{t\geqslant 0}$ so that suitable results on sectorial operators could be applied. We refer the reader to [16] and [27] for the general theory on analytic semigroups. To be precise, we will assume the following:

Hypothesis 4. The semigroup $\{T_0(t)\}_{t\geqslant 0}$ is an analytic semigroup on X. Moreover, the infinitesimal generator A has compact resolvent.

The direct sum decomposition (3) allows us to assure that for each $\beta \geqslant 0$, the fractional power operator $(-A_S)^\beta$ can be defined on a domain $X_S^\beta \subset X$ which is a Banach space with respect to the norm $\|\varphi_S\|_\beta := \|(-A_S)^\beta \varphi\|_X$.

To facilitate the reading, we summarise in the following theorem the main theoretical results that we will need (see [16, Th. 1.4.8], [27, Th. 6.1.3]):

Theorem 3. *For each $\beta \geqslant 0$, $t > 0$, the following hold*:

i) *There exists a constant $C_\beta > 0$ such that $\forall \varphi_S \in X_S^\beta$, $\|\varphi_S\|_X \leqslant C_\beta \|\varphi_S\|_\beta$. Moreover, for $\beta > 0$, the embedding $X_S^\beta \subset S$ is compact.*

ii) *$T_{\varepsilon S}(t)(S) \subset X_S^\beta$ and $T_{\varepsilon S}(t)(-A_S)^\beta \varphi_S = (-A_S)^\beta T_{\varepsilon S}(t)\varphi_S$, $\varphi_S \in X_S^\beta$.*

iii) *$(-A_S)^\beta T_{\varepsilon S}(t)\varphi_S$ is a bounded linear operator on S and moreover*

$$\left\|(-A_S)^\beta T_{\varepsilon S}(t)\varphi_S\right\|_X \leqslant M_\beta \left(\frac{t}{\varepsilon}\right)^{-\beta} e^{-\alpha^* t/\varepsilon} \|\varphi_S\|_X.$$

If $n_0 \in X$ is an initial condition such that the corresponding solution to $(\mathrm{CP})_\varepsilon$, $n_\varepsilon(t;n_0)$, exists on $[0,+\infty)$, we define the *positive orbit* as $\gamma_+^{(\varepsilon)}(n_0) := \{n_\varepsilon(t;n_0);\ t \geqslant 0\}$ and also, for a set $B \subset X$ of such initial conditions, $\gamma_+^{(\varepsilon)}(B) := \bigcup_{n_0 \in B} \gamma_+^{(\varepsilon)}(n_0)$.

Then we have the following:

Proposition 2. *Keeping the hypotheses and notations of Proposition* 1 *and assuming Hypothesis* 4, *for each $\beta \in (0,1)$, $\varepsilon \in (0,\varepsilon^*)$, the set $\gamma_+^{(\varepsilon)}(\mathcal{N}^\beta(W^*(\mathcal{A});\delta^*))$ is a precompact set in X. We have introduced the notation*:

$$\mathcal{N}^\beta\big(W^*(\mathcal{A});\delta^*\big) := \big\{n_0 = N_0\mu + \rho_0;\ N_0 \in W^*(\mathcal{A}),\ \rho_0 \in X_S^\beta,\ C_\beta\|\rho_0\|_\beta < \delta^*\big\}$$

where $C_\beta > 0$ is the constant mentioned in Theorem 3(i).

Proof. It is evident that $\mathcal{A}\mu \subset \mathcal{N}^\beta(W^*(\mathcal{A});\delta^*) \subset \mathcal{N}(W^*(\mathcal{A});\delta^*)$, and therefore for each initial value $n_0 \in \mathcal{N}^\beta(W^*(\mathcal{A});\delta^*)$, Proposition 1 assures that the positive orbit $\gamma_+(n_0)$ is defined. Moreover $N_\varepsilon(t;n_0) \in U_0(\mathcal{A})$ and thus there exists a constant $R_0 > 0$ such that

$$\sup\big\{\big|N_\varepsilon(t;n_0)\big|,\ t \geqslant 0,\ n_0 \in \mathcal{N}^\beta\big(W^*(\mathcal{A});\delta^*\big)\big\} \leqslant R_0.$$

On the other hand, since $\rho_0 \in X_S^\beta$, from Theorem 3(ii)–(iii) and (5)–(6), we have for all $\beta \in (0,1)$:

$$\begin{aligned}\big\|\rho_\varepsilon(t;n_0)\big\|_\beta &\leqslant \big\|T_{\varepsilon S}(t)(-A_S)^\beta \rho_0\big\|_X + M_\beta \int_0^t \big[(t-\sigma)/\varepsilon\big]^{-\beta} e^{-\alpha^*(t-\sigma)/\varepsilon} \big\|\mathcal{F}_S\big(n_\varepsilon(\sigma;n_0)\big)\big\|_X\, d\sigma \\ &\leqslant M_S e^{-\alpha^* t/\varepsilon}\|\rho_0\|_\beta + M_\beta\big(C_1 + C_2\delta^*\big)\int_0^t \big[(t-\sigma)/\varepsilon\big]^{-\beta} e^{-\alpha^*(t-\sigma)/\varepsilon}\, d\sigma \\ &\leqslant C_{1\beta}\delta^* + C_{2\beta}\varepsilon^* + C_{3\beta}\varepsilon^*\delta^*\end{aligned}$$

which means that for each $\beta \in (0,1)$ there exists a constant $R^* > 0$ such that $\forall \varepsilon \in (0,\varepsilon^*)$,

$$\sup\big\{\big\|\rho_\varepsilon(t;n_0)\big\|_\beta;\ t \geqslant 0,\ n_0 \in \mathcal{N}^\beta\big(W^*(\mathcal{A});\delta^*\big)\big\} \leqslant R^*.$$

Then, the precompactness of $\gamma_+^{(\varepsilon)}(\mathcal{N}^\beta(W^*(\mathcal{A});\delta^*))$ is an immediate consequence of Theorem 3(i). □

We can now establish the main result of this section:

Theorem 4. *Under Hypotheses* 1, 2, 3, 4, *assume that there exists a compact attractor $\mathcal{A}$ for the aggregated model* (4). *Then, there exists $\varepsilon_0^* > 0$ such that $\forall \varepsilon \in (0,\varepsilon_0^*)$, there exists a compact attractor $\mathcal{A}_\varepsilon$ for the perturbed model* $(\mathrm{CP})_\varepsilon$. *Moreover we have $\mathcal{A}_\varepsilon \subset \mathcal{N}(W(\mathcal{A});\delta)$ for each neighbourhood of $\mathcal{A}\mu$ in X and $\varepsilon > 0$ small enough. Also* $\operatorname{diam}(S \cap \mathcal{A}_\varepsilon) \to 0$ $(\varepsilon \to 0_+)$.

Proof. It will be a direct consequence of Propositions 1 and 2. Let us define $\mathcal{A}_\varepsilon$ as the omega-limit set of the set $\mathcal{N}^\beta(W^*(\mathcal{A});\delta^*)$ introduced in Proposition 2. As an immediate consequence of this proposition (see [12, Lemma 3.1.2]), we can assure that $\mathcal{A}_\varepsilon$ is a nonempty, connected and compact set in X which is invariant with respect to $(\mathrm{CP})_\varepsilon$ for $\varepsilon \in (0,\varepsilon^*)$ and attracts $\mathcal{N}^\beta(W^*(\mathcal{A});\delta^*)$. Set $n_\varepsilon^* = N_\varepsilon^*\mu + \rho_\varepsilon^* \in \mathcal{A}_\varepsilon$, for $\varepsilon \in (0,\varepsilon^*)$. Straightforward calculations show that there exists a constant $C(\delta^*;\beta) > 0$ such that $\|\rho_\varepsilon^*\|_\beta \leqslant \varepsilon C(\delta^*;\beta)$ and also that $N_\varepsilon^* \in W^*(\mathcal{A})$. Therefore, choosing $\varepsilon_0^* \leqslant \min(\varepsilon^*, \delta^*/(C_\beta C(\delta^*;\beta)))$, we have for each $\varepsilon \in (0,\varepsilon_0^*)$ that $\|\rho_\varepsilon^*\|_X \leqslant C_1\varepsilon$ and also that $\mathcal{A}_\varepsilon \subset \mathcal{N}^\beta(W^*(\mathcal{A});\delta^*)$. This means that $\mathcal{A}_\varepsilon$ is a compact attractor for $(\mathrm{CP})_\varepsilon$ and moreover that $\operatorname{diam}(S \cap \mathcal{A}_\varepsilon) \to 0$ when $\varepsilon \to 0_+$, as we wanted to prove. □

We conclude this section by considering a particular case in which $\ker A$ is invariant under operator $\mathcal{F}$. This assumption allows us to improve the general result established in Theorem 4. To be precise, we have the following:

Corollary 1. *Under hypotheses of Theorem 4, let us assume that $\mathcal{F}(\ker A) \subset \ker A$. Then, for all $\varepsilon \in (0, \varepsilon_0^*)$, $\mathcal{A}\mu$ is a compact attractor for the perturbed model $(\mathrm{CP})_\varepsilon$.*

Proof. First of all, notice that $\mathcal{A}\mu$ is an invariant set for $(\mathrm{CP})_\varepsilon$. On the other hand, notice that estimation (6) can be substituted by

$$\left\| \mathcal{F}_S\big(n_\varepsilon(s)\big)\right\|_X = \left\| \mathcal{F}_S\big(N_\varepsilon(s)\mu + \rho_\varepsilon(s)\big) - \mathcal{F}_S\big(N_\varepsilon(s)\mu\big)\right\|_X \leqslant C_1 \left\|\rho_\varepsilon(s)\right\|_X$$

which in turn, modifies the estimation (ii) in Proposition 1 giving:

$$\text{(ii)}^* \quad \left\|\rho_\varepsilon(t)\right\|_X \leqslant C_1 e^{-\beta t/\varepsilon} \|\rho_0\|_X$$

and therefore $\mathcal{A}_\varepsilon \subset \ker A$.

Moreover, and referring to the solutions considered in Proposition 1, the component $N_\varepsilon(t)$ satisfies an asymptotically autonomous O.D.E.:

$$N'_\varepsilon(t) = \big\langle \mu^*, \mathcal{F}\big(n_\varepsilon(t)\big)\big\rangle = \big\langle \mu^*, \mathcal{F}\big(N_\varepsilon(t)\mu\big)\big\rangle + \mathcal{G}\big(t, N_\varepsilon(t)\big)$$

where

$$\begin{aligned}\big|\mathcal{G}\big(t, N_\varepsilon(t)\big)\big| &= \big|\big\langle \mu^*, \mathcal{F}\big(n_\varepsilon(t)\big) - \mathcal{F}\big(N_\varepsilon(t)\mu\big)\big\rangle\big| \leqslant C_1 \left\|\mathcal{F}\big(n_\varepsilon(t)\big) - \mathcal{F}\big(N_\varepsilon(t)\mu\big)\right\|_X \\ &\leqslant C_1 \left\|\rho_\varepsilon(t)\right\|_X \leqslant C_1 e^{-\beta t/\varepsilon} \to 0 \quad (t \to +\infty).\end{aligned}$$

Then (see [11,31]) $\mathcal{A}_\varepsilon$ is a union of invariant sets for the aggregated model, which belong to $\mathcal{N}^\beta(W^*(\mathcal{A}); \delta^*)$. But all such invariant sets must belong to $\mathcal{A}\mu$, what completes the proof. □

3. Two-time scales in reaction–diffusion models of population dynamics

In this section we illustrate the general aggregation of variables method described in the previous section by applying it to a reaction–diffusion system which represents the dynamics of several continuously spatially distributed populations whose evolution processes occur at two different time scales: a slow one for the demography and a fast one for migrations. Let us consider q $(q \geqslant 1)$ populations living in a spatial region $\Omega \subset \mathbf{R}^p$ $(p \geqslant 1)$, where Ω is a nonempty bounded, open and connected set with *smooth* boundary $\partial\Omega \in C^k$, $k \geqslant 1$. Let $n_i(x,t)$, $i = 1, \ldots, q$, be their spatially structured population densities i.e., $\int_{\Omega_0} n_i(x,t)\,dx$ represents the number of individuals of population i that at time t are occupying the region $\Omega_0 \subset \Omega$ and set $n(x,t) := (n_1(x,t), \ldots, n_q(x,t))^T$.

We assume that the demography is given by a nonlinear reaction term $f(x,n)$ that satisfies the following regularity conditions:

Hypothesis 5. The function $f : \overline{\Omega} \times \mathbf{R}^q \longrightarrow \mathbf{R}^q$, $f := (f_1, \ldots, f_q)$, is continuous and satisfies the following:

There exists a real-valued continuous positive function h defined on $\overline{\Omega} \times \mathbf{R}^q \times \mathbf{R}^q$ such that $\forall x \in \overline{\Omega}$ and $\forall u, v \in \mathbf{R}^q$:

$$\big|f(x,u) - f(x,v)\big| \leqslant h(x,u,v)|u - v|.$$

We also assume a linear diffusion process in Ω for each population, with coefficient $D_i \in C^2(\overline{\Omega})$, $D_i(x) \geqslant d_i^* > 0$, $i = 1, \ldots, q$, that occurs at a fast time scale determined by a parameter $\varepsilon > 0$ small enough. A standard application of the balance law leads to the following two-time reaction–diffusion system for the population densities, with $i = 1, \ldots, q$:

$$\frac{\partial n_i}{\partial t}(x,t) = \frac{1}{\varepsilon} \operatorname{div}\big(D_i(x) \operatorname{grad} n_i(x,t)\big) + f_i\big(x, n(x,t)\big), \quad x \in \Omega,\ t > 0 \tag{10}$$

completed with Neumann boundary conditions:

$$\frac{\partial n_i}{\partial \nu}(x,t) = 0, \quad x \in \partial\Omega,\ t > 0 \tag{11}$$

which indicates that the spatial domain is isolated from the external environment, plus initial conditions:

$$n(x,0) = n_0(x), \quad x \in \Omega,\ n_0(x) := \big(n_1^0(x), \ldots, n_q^0(x)\big)^T. \tag{12}$$

Rescaling the time as $t = \varepsilon\tau$ in Eqs. (10) and making $\varepsilon \to 0_+$, we obtain the dynamics at a fast time scale:

$$\frac{\partial n_i}{\partial \tau}(x,\tau) = \operatorname{div}\big(D_i(x)\operatorname{grad} n_i(x,\tau)\big), \quad i = 1,\dots,q.$$

At this fast time scale the total population $N_i(\tau) := \int_\Omega n_i(x,\tau)\,dx$ satisfies that

$$N_i'(\tau) = \int_\Omega \operatorname{div}\big(D_i(x)\operatorname{grad} n_i(x,\tau)\big)\,dx = \int_{\partial\Omega} D_i(x)\frac{\partial n_i}{\partial \nu}(x,\tau)\,d\sigma = 0$$

which reflects the obvious result that the total population is conserved under the migration process, without taking into account the demographic evolution. This simple idea suggests constructing a reduced model to approximate the model (10), (11), (12), taking as *global variables* the total populations:

$$N_i(t) := \int_\Omega n_i(x,t)\,dx; \qquad N(t) := \big(N_1(t),\dots,N_q(t)\big)^T.$$

Integrating with respect to the space variable x on both sides of Eqs. (10), applying the Gauss Theorem and bearing in mind the Neumann boundary conditions (11), we have

$$N_i'(t) = \int_\Omega f_i\big(x, n(x,t)\big)\,dx, \quad i = 1,\dots,q. \tag{13}$$

Notice that the right-hand side of Eq. (13) is expressed in terms of the density $n(x,t)$. To avoid this difficulty, we will look for an approximation of $n(x,t)$ in terms of the total populations. To this end, we assume that the fast dynamics reach an equilibrium. Recall that the only equilibria of the fast dynamics are the constants and since the total population is conserved under the fast dynamics, the initial conditions (12) fix the values of the stationary states for the fast dynamics:

$$\int_\Omega n_i^0(x)\,dx = n_i^* \operatorname{vol}(\Omega) \quad \Longrightarrow \quad n_i^* = \frac{1}{\operatorname{vol}(\Omega)}\int_\Omega n_i^0(x)\,dx, \quad i = 1,\dots,q$$

where $\operatorname{vol}(\Omega)$ is the Lebesgue measure of the domain Ω. That is, in absence of demography, the stationary state of the population is a homogeneous distribution on the spatial region.

Then, coming back to the construction of an approximated model for the dynamics of the total population, the above considerations suggest the following approximation:

$$n_i(x,t) \approx \frac{N_i(t)}{\operatorname{vol}(\Omega)}, \quad i = 1,\dots,q$$

which yields the *aggregated model* of (10), (11), (12):

$$N'(t) = F\big(N(t)\big), \qquad N(0) = N_0 := \int_\Omega n_0(x)\,dx \tag{14}$$

where $F : \mathbf{R}^q \longrightarrow \mathbf{R}^q$, $F := (F_1,\dots,F_q)$ is the function defined by

$$\forall u \in \mathbf{R}^q, \quad F(u) := \int_\Omega f\left(x, \frac{u}{\operatorname{vol}(\Omega)}\right)dx.$$

To apply the comparison result between both models given by Theorem 4, we have to formulate (10), (11), (12) as an abstract evolution equation in the setting of the C_0-semigroup theory. To this end, we choose as state space $E := C(\overline{\Omega})$, the Banach space of continuous real-valued functions defined on $\overline{\Omega}$, endowed with the supremum norm $\|\varphi\|_\infty := \sup_{x\in\overline{\Omega}}|\varphi(x)|$, so that $X := [C(\overline{\Omega})]^q$ endowed with the product norm.

We need to specify the conditions that assure that the linear diffusion operator in Eqs. (10) together with a Neumann boundary condition is the infinitesimal generator of a C_0-semigroup on $C(\overline{\Omega})$. According to [22], let us consider the linear operator $\hat{A}_i$ defined by

$$(\hat{A}_i\varphi)(x) := \operatorname{div}\big(D_i(x)\operatorname{grad}\varphi(x)\big); \quad x \in \Omega$$

with domain:

$$D(\hat{A}_i) := \left\{\varphi \in C^2(\overline{\Omega});\ \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|D^m\varphi(x) - D^m\varphi(y)|}{\|x-y\|^\alpha} < +\infty,\ \frac{\partial\varphi}{\partial\nu} = 0 \text{ in } \partial\Omega\right\}$$

where $0 < \alpha < 1$ and $D^m\varphi$, $m := (m_1,\dots,m_N)$ represents the partial derivatives of φ of order $2 = m_1 + \cdots + m_N$.

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Notice that $\hat{A}_i$ is a strongly elliptic operator.

We refer to [22,23,27] for the general theory which provides the main results that we need, explained in the following proposition.

Proposition 3. *(See [22].) The operator A_i defined as the closure in $C(\overline{\Omega})$ of operator $\hat{A}_i$ is the infinitesimal generator of an analytic and compact C_0-semigroup $\{T_{0i}(t)\}_{t\geqslant 0}$ on $C(\overline{\Omega})$.*

Moreover, the semigroup $\{T_{0i}(t)\}_{t\geqslant 0}$ satisfies the following:

i) *$T_{0i}(t)(C_+(\overline{\Omega})) \subset C_+(\overline{\Omega})$, $t \geqslant 0$, where $C_+(\overline{\Omega})$ is the subset of real-valued nonnegative continuous functions on $\overline{\Omega}$ (see [33,28]).*
ii) *$\{T_{0i}(t)\}_{t\geqslant 0}$ is an irreducible semigroup on $C(\overline{\Omega})$ (see [23]).*

Finally, let us introduce the operator $\mathcal{F}: X \longrightarrow X$ defined by

$$\forall \varphi := (\varphi_1, \ldots, \varphi_q)^T \in X, \quad \mathcal{F}(\varphi)(x) := f\big(x, \varphi(x)\big), \quad x \in \overline{\Omega}. \tag{15}$$

Bearing in mind the above definition of operator $A := \operatorname{diag}(A_1, \ldots, A_q)$ and making the usual identification $n(\cdot, t) := n(t)(\cdot)$ we can write (10), (11), (12) as the abstract Cauchy problem $(\mathrm{CP})_\varepsilon$ of the previous section, on the Banach space X. An immediate consequence of Hypothesis 5 is that operator $\mathcal{F}$ satisfies Hypothesis 2 and Proposition 3 assures that operator A satisfies all the assumptions of Theorem 4 and thus its conclusion holds, providing an approximation result for the behaviour of solutions to the initial two-time reaction–diffusion model through the solutions to the aggregated model.

As a simple illustration, we apply the above ideas to a predator-prey model with fast constant diffusion and population growth of the preys given by a logistic law. To be precise, we are considering the model:

$$(\mathrm{FPP}) \quad \begin{cases} \dfrac{\partial n}{\partial t}(x,t) = \dfrac{D_n}{\varepsilon}\Delta n(x,t) + r(x)n(x,t)\left[1 - \dfrac{n(x,t)}{K(x)}\right] - a(x)n(x,t)p(x,t), \\ \dfrac{\partial p}{\partial t}(x,t) = \dfrac{D_p}{\varepsilon}\Delta p(x,t) + b(x)n(x,t)p(x,t) - \mu(x)p(x,t) \end{cases}$$

with $x \in \Omega$, $t > 0$. The model is completed with Neumann boundary conditions:

$$\frac{\partial n}{\partial \nu}(x,t) = \frac{\partial p}{\partial \nu}(x,t) = 0, \quad x \in \partial\Omega,\ t > 0$$

and initial conditions:

$$n(x,0) = n_0(x), \qquad p(x,0) = p_0(x), \quad x \in \Omega$$

where $n(x,t)$ and $p(x,t)$ represent the population densities of preys and predators respectively.

The global variables are the total populations of predators and preys:

$$N(t) := \int_\Omega n(x,t)\,dx; \qquad P(t) := \int_\Omega p(x,t)\,dx$$

which satisfy the following system, which is obtained by integrating on Ω on both sides of system (FPP):

$$\begin{cases} N'(t) = \displaystyle\int_\Omega r(x)n(x,t)\left(1 - \frac{n(x,t)}{K(x)}\right)dx - \int_\Omega a(x)n(x,t)p(x,t)\,dx, \\ P'(t) = \displaystyle\int_\Omega b(x)n(x,t)p(x,t)\,dx - \int_\Omega \mu(x)p(x,t)\,dx. \end{cases} \tag{16}$$

Since the right-hand side of Eqs. (16) is expressed in terms of the densities $n(x,t)$, $p(x,t)$, we make the approximation:

$$n(x,t) \approx \frac{N(t)}{\operatorname{vol}(\Omega)}; \qquad p(x,t) \approx \frac{P(t)}{\operatorname{vol}(\Omega)}$$

which leads to:

$$(\mathrm{AM}) \quad \begin{cases} N'(t) = r^* N(t)\left(1 - \dfrac{N(t)}{K^*}\right) - a^* N(t)P(t), \\ P'(t) = b^* N(t)P(t) - \mu^* P(t) \end{cases}$$

where

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$$r^* := \frac{1}{\mathrm{vol}(\Omega)} \int_\Omega r(x)\,dx; \qquad K^* := \frac{\mathrm{vol}(\Omega) \int_\Omega r(x)\,dx}{\int_\Omega (r(x)/K(x))\,dx};$$

$$a^* := \frac{1}{(\mathrm{vol}(\Omega))^2} \int_\Omega a(x)\,dx; \qquad b^* := \frac{1}{(\mathrm{vol}(\Omega))^2} \int_\Omega b(x)\,dx;$$

$$\mu^* := \frac{1}{\mathrm{vol}(\Omega)} \int_\Omega \mu(x)\,dx.$$

Summing up, the aggregated model is a classical predator-prey model with logistic grow for the prey, in which the initial spatial structure has been taken into account in the parameters.

If $\mu^* < b^* K^*$, direct calculations show that (AM) has a unique locally asymptotically stable positive equilibrium $P^* := (\mu^*/b^*, (r^*/a^*)(1 - \mu^*/(b^* K^*)))$ and then, for $\varepsilon > 0$ small enough, (FPP) has a compact local attractor $\mathcal{A}_\varepsilon$ close to P^*.

3.1. Positive solutions to the two-time reaction–diffusion model: a comparison result

In this section we will proceed to compare when $\varepsilon \to 0_+$ the solutions to (10), (11), (12), or its abstract formulation $(\mathrm{CP})_\varepsilon$, with the solutions to the aggregated model (14) corresponding to the same initial data and without assuming the existence of equilibria for the aggregated model. To this end we have to ensure the global existence of these solutions as well as the existence of suitable bounds for the component $\rho_\varepsilon(t)$, which can be done assuming additional conditions for the reaction term given by function f. In a general situation, a sufficient condition consists of assuming the existence of a compact invariant region for the perturbed reaction–diffusion equation (see [11] and [7]). In this work we present a slightly different situation, restricting our analysis to the comparison of positive solutions and to simplify we consider a scalar formulation of both problems.

To be precise, in addition to Hypothesis 5 we will assume the following smoothness assumption, that is a standard sufficient condition to eliminate blow-up of positive solutions to $(\mathrm{CP})_\varepsilon$:

Hypothesis 6. The function $f : \overline{\Omega} \times \mathbf{R} \longrightarrow \mathbf{R}$ satisfies the following:

i) $f(x, 0) = 0$, $\forall x \in \overline{\Omega}$.
ii) There exists a constant $C > 0$ such that $\forall x \in \overline{\Omega}$ and $\forall u \in \mathbf{R}$ with $|u| \geqslant C$, we have $f(x, u) \leqslant 0$.

- **Existence and boundedness of global positive solutions to both problems**

Positivity of solutions to (14) is an immediate consequence of the uniqueness of solutions together with $F(0) = 0$. That is, if $N_0 > 0$, then $N(t) > 0$ for all $t \in [0, w_+)$. Now let us show that positive solutions are defined on $[0, +\infty)$ and are globally bounded by some constant depending on the initial data.

Lemma 1. *Assume Hypotheses* 5 *and* 6*. Then, for each initial data* $N_0 > 0$*, the corresponding solution* $N(t)$ *to* (14) *is defined on* $[0, +\infty)$ *and there exists a constant* $K(N_0) > 0$ *such that* $0 < N(t) \leqslant K(N_0)$ *for all* $t \in [0, +\infty)$.

Proof. The proof is based on standard arguments closely related to the fact that the aggregated model is an autonomous scalar differential equation. First of all, notice that Hypothesis 6(ii) leads to the existence of a constant $C^* > 0$ such that $|u| \geqslant C^*$ implies that $F(u) \leqslant 0$.

a) Set $N_0 \geqslant C^*$, so that $F(N_0) \leqslant 0$. If $F(N_0) = 0$, then N_0 is an equilibrium point of the aggregated model and there is nothing to prove. Therefore, assume that $F(N_0) < 0$ and define $t^* := \min\{t > 0;\ N(s) \leqslant N_0,\ \forall s \in (0, t)\}$.

If $w_+ < +\infty$, then $t^* < +\infty$ and $N(t^*) = N_0$ so that $N'(t^*) = F(N(t^*)) = F(N_0)$. This means that there exists $\delta > 0$ such that for all $t \in (t^*, t^* + \delta)$ with $t^* + \delta < w_+$, we have $N(t) < N(t^*) = N_0$, which is a contradiction. Consequently, $w_+ = +\infty$ and also $0 < N(t) \leqslant N_0$ for all $t \in [0, +\infty)$.

b) Set $0 < N_0 < C^*$. Then, defining $t^* := \min\{t > 0;\ N(s) < C^*,\ \forall s \in (0, t)\}$, we can assure that $t^* \in (0, w_+)$. If $w_+ < +\infty$, then $t^* < +\infty$ and $N(t^*) = C^*$ which, as in the previous case, leads to the contradiction $N'(t^*) = F(N(t^*)) = F(C^*) \leqslant 0$. That is, $w_+ = +\infty$ and also $0 < N(t) < C^*$ for all $t \geqslant 0$.

Choosing $K(N_0) := \max(N_0, C^*) > 0$, the lemma is proved. □

Now we concentrate on nonnegative solutions to $(\mathrm{CP})_\varepsilon$: we will prove a monotonicity result and global existence of nonnegative solutions. Our arguments will follow closely those given in [8,14,15], for general reaction–diffusion equations.

Proposition 4. *If the initial data* $n_0 \in C(\overline{\Omega})$ *is nonnegative, then the corresponding maximal solution* n_ε *satisfies that* $n_\varepsilon(x, t) \geqslant 0$, $\forall x \in \Omega$, $\forall t \in [0, T_{\max})$.

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Proof. First of all, let us observe that applying a Green formula and bearing in mind the Neumann boundary condition, we have $\forall \varphi \in D(\hat{A})$:

$$\int_\Omega \operatorname{div}\big(D(x)\operatorname{grad}\varphi(x)\big)\varphi^-(x)\,dx = -\int_\Omega D(x)\operatorname{grad}\varphi(x)\operatorname{grad}\varphi^-(x)\,dx$$
$$= \int_\Omega D(x)\big\|\operatorname{grad}\varphi^-(x)\big\|^2\,dx \geqslant 0 \tag{17}$$

where we have introduced the notation:

$$\forall \varphi \in C(\overline{\Omega}), \quad \varphi^+(x) := \max\big(\varphi(x), 0\big); \qquad \varphi^-(x) := -\min\big(0, \varphi(x)\big).$$

Multiplying in both sides of $(\mathrm{CP})_\varepsilon$ by n_ε^- and integrating on Ω we have

$$\int_\Omega \frac{\partial n_\varepsilon}{\partial t}(x,t)n_\varepsilon^-(x,t)\,dx = \frac{1}{\varepsilon}\int_\Omega An_\varepsilon(x,t)n_\varepsilon^-(x,t)\,dx + \int_\Omega \mathcal{F}\big(n_\varepsilon(t)\big)(x)n_\varepsilon^-(x,t)\,dx.$$

Bearing in mind that $n_\varepsilon(x,t) = n_\varepsilon^+(x,t) - n_\varepsilon^-(x,t)$ and the estimation (17), we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \big[n_\varepsilon^-(x,t)\big]^2\,dx \leqslant -\int_\Omega \mathcal{F}\big(n_\varepsilon(t)\big)(x)n_\varepsilon^-(x,t)\,dx \leqslant \int_\Omega \big|\mathcal{F}\big(n_\varepsilon(t)\big)(x)\big|\big|n_\varepsilon^-(x,t)\big|\,dx.$$

Fix $T \in (0, T_{\max})$, $T < +\infty$, so that $\forall t \in [0,T]$:

$$\big|n_\varepsilon(x,t)\big| \leqslant \sup_{t\in[0,T]}\big\|n_\varepsilon(\cdot,t)\big\|_\infty := \gamma(T) < +\infty.$$

This estimation together with Hypotheses 5 and 6(i) yield, $\forall t \in [0,T]$.

$$\big|\mathcal{F}\big(n_\varepsilon(t)\big)(x)\big| = \big|f\big(x, n_\varepsilon(x,t)\big)\big| = \big|f\big(x, n_\varepsilon(x,t)\big) - f(x,0)\big|$$
$$\leqslant R(T)\big|n_\varepsilon(x,t)\big|$$

for some constant $R(T) > 0$.

Let us simplify by introducing the notation $v(t) := \int_\Omega [n_\varepsilon^-(x,t)]^2\,dx$. Then we have for $t \in [0,T]$:

$$\frac{1}{2}v'(t) \leqslant R(T)\int_\Omega \big|n_\varepsilon(x,t)\big|\big|n_\varepsilon^-(x,t)\big|\,dx = R(T)v(t)$$

which leads to $0 \leqslant v(t) \leqslant v(0)e^{2R(T)t}$. The nonnegativity of the initial data n_0 implies that $v(0) = 0$, which in turn implies that $n_\varepsilon^-(x,t) = 0$, $\forall t \in [0,T]$. This equality holds for all $T \in (0, T_{\max})$. That is, we conclude that $n_\varepsilon(x,t) \geqslant 0$, $\forall (x,t) \in \Omega \times [0, T_{\max})$ as we wanted to prove. □

A straightforward modification of the above arguments leads to the following monotonicity result:

Corollary 2. *Let n_ε, v_ε be two continuous solutions to* $(\mathrm{CP})_\varepsilon$ *both defined in an interval* $[0,T]$ $(T > 0)$*, corresponding to initial data $n_0, v_0 \in C(\overline{\Omega})$ such that $n_0(x) \geqslant v_0(x)$, $x \in \Omega$. Then,*

$$\forall x \in \Omega,\ \forall t \in [0,T], \quad n_\varepsilon(x,t) \geqslant v_\varepsilon(x,t).$$

Proof. The function $w(x,t) := n_\varepsilon(x,t) - v_\varepsilon(x,t)$ satisfies the following equation, together with a nonnegative initial data $w_0(x) := n_0(x) - v_0(x) \geqslant 0$:

$$w'(t) = \frac{1}{\varepsilon}Aw(t) + \mathcal{F}\big(n_\varepsilon(t)\big) - \mathcal{F}\big(v_\varepsilon(t)\big).$$

Multiplying in both sides of that equation by $w^-(x,t)$ and integrating on Ω, similar calculations to those in Proposition 4 lead to the following inequalities:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \big|w^-(x,t)\big|^2\,dx \leqslant -\int_\Omega \big|\mathcal{F}\big(n_\varepsilon(t)\big)(x) - \mathcal{F}\big(v_\varepsilon(t)\big)(x)\big|\big|w^-(x,t)\big|\,dx$$
$$\leqslant R^*(T)\int_\Omega \big|w^-(x,t)\big|^2\,dx$$

for some constant $R^*(T) > 0$.

Therefore

$$\forall t \in [0,T], \quad \int_\Omega \left|w^-(x,t)\right|^2 dx \leqslant \left(\int_\Omega \left|w^-(x,0)\right|^2 dx\right) e^{2R^*(T)t} = 0$$

which implies that $w(x,t) \geqslant 0$, $\forall (x,t) \in \Omega \times [0,T]$ as we wanted to prove. □

As a consequence we can prove the following result concerning the existence of global solutions:

Proposition 5. *The continuous solutions to* $(\mathrm{CP})_\varepsilon$ *corresponding to nonnegative continuous initial data are defined on* $[0,+\infty)$ *and are uniformly bounded on t and on* $\varepsilon > 0$.

Proof. Let $n_0 \in C(\overline{\Omega})$, $n_0 \geqslant 0$ be an initial data to $(\mathrm{CP})_\varepsilon$, let $n_\varepsilon(\cdot,t)$ be the corresponding nonnegative maximal solution defined on $[0,T_{\max})$, and set $K_0(n_0) > \max(C, \|n_0\|_\infty) > 0$, where $C > 0$ is the constant mentioned in Hypothesis 6(ii).

Multiplying in both sides of $(\mathrm{CP})_\varepsilon$ by $(n_\varepsilon(\cdot,t) - K_0(n_0))^+$ and integrating on Ω, similar calculations to those in the proof of Proposition 4 lead to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left|\left(n_\varepsilon(x,t) - K_0(n_0)\right)^+\right|^2 dx \leqslant \int_\Omega \mathcal{F}\big(n_\varepsilon(t)\big)(x)\big(n_\varepsilon(x,t) - K_0(n_0)\big)^+ dx.$$

Since

$$\begin{aligned} 0 \leqslant n_\varepsilon(x,t) \leqslant K_0(n_0) \quad &\Longrightarrow \quad \big(n_\varepsilon(x,t) - K_0(n_0)\big)^+ = 0, \\ n_\varepsilon(x,t) \geqslant K_0(n_0) \quad &\Longrightarrow \quad \mathcal{F}\big(n_\varepsilon(t)\big)(x) \leqslant 0 \end{aligned}$$

we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left|\left(n_\varepsilon(x,t) - K_0(n_0)\right)^+\right|^2 dx \leqslant 0$$

which gives $(n_\varepsilon(x,t) - K_0(n_0))^+ = 0$ and then

$$\forall (x,t) \in \Omega \times [0,T_{\max}), \quad 0 \leqslant n_\varepsilon(x,t) \leqslant K_0(n_0).$$

This implies that $T_{\max} = +\infty$ and $\sup_{t\geqslant 0}\|n_\varepsilon(\cdot,t)\|_\infty \leqslant K_0(n_0)$, as we wanted to prove. □

- **Comparison between the solutions to the perturbed and aggregated models**

Our main goal will be to establish an approximation result for the solutions to the perturbed model in terms of that of the aggregated model, for $\varepsilon > 0$ small enough. To this end, let us consider a solution to $(\mathrm{CP})_\varepsilon$, $n_\varepsilon(t) := N_\varepsilon(t) + \rho_\varepsilon(t)$, corresponding to a nonnegative initial data $n_0 \in C(\overline{\Omega})$. Since n_ε is nonnegative, defined on $[0,+\infty)$ and uniformly bounded with respect to t and ε, we can introduce in the variation of constants formula (5) the bound:

$$\left\|\mathcal{F}_S\big(n_\varepsilon(\sigma)\big)\right\|_\infty \leqslant C_1\left\|\mathcal{F}\big(n_\varepsilon(\sigma)\big) - \mathcal{F}(0)\right\|_\infty \leqslant C_2(n_0)\left\|n_\varepsilon(\sigma)\right\|_\infty \leqslant C_3(n_0)$$

and then, a straightforward calculation provides the following estimation, valid for $t \geqslant 0$, $\varepsilon > 0$:

$$\left\|\rho_\varepsilon(t)\right\|_\infty \leqslant C_1 e^{-(\alpha^*/\varepsilon)t}\|\rho_0\|_\infty + C_2(n_0)\varepsilon. \tag{18}$$

The approximation result between the solutions to $(\mathrm{CP})_\varepsilon$ and the aggregated model (14) we are looking for in this setting consists of comparing the solutions to both problems corresponding to the nonnegative common initial data n_0. To this end, notice that $N_\varepsilon(t)$ satisfies an O.D.E. that can be written as a perturbation of the aggregated model (14):

$$N_\varepsilon'(t) = F\big(N_\varepsilon(t)\big) + \mathcal{G}_\varepsilon(t) \tag{19}$$

where

$$\mathcal{G}_\varepsilon(t) := \int_\Omega \left[f\left(x, \frac{N_\varepsilon(t)}{\mathrm{vol}(\Omega)} + \rho_\varepsilon(t)(x)\right) - f\left(x, \frac{N_\varepsilon(t)}{\mathrm{vol}(\Omega)}\right)\right] dx.$$

We will proceed to estimate $y_\varepsilon(t) := N_\varepsilon(t) - N(t)$.

Since Lemma 1 and Proposition 5 establish the bounds:

$$\forall t \geqslant 0,\ \forall x \in \Omega, \quad \left|N(t)\right| \leqslant K(N_0), \qquad \left|n_\varepsilon(x,t)\right| \leqslant K_0(n_0)$$

bearing in mind the local Lipschitz continuity of operator $\mathcal{F}$ we have, for $t > 0$:

$$\left|y_\varepsilon(t)\right| \leqslant \int_0^t \left(\int_\Omega \left| f\left(x, \frac{N_\varepsilon(\sigma)}{\text{vol}(\Omega)} + \rho_\varepsilon(\sigma)(x)\right) - f\left(x, \frac{N(\sigma)}{\text{vol}(\Omega)}\right)\right| dx\right) d\sigma$$
$$\leqslant H_0(n_0)\left[\int_0^t \left|y_\varepsilon(\sigma)\right| d\sigma + \int_0^t \left(\int_\Omega \left|\rho_\varepsilon(\sigma)(x)\right| dx\right) d\sigma\right]$$

for some constant $H_0(n_0) > 0$.

On the other hand, bearing in mind the estimation (18), we have

$$\int_0^t \left(\int_\Omega \left|\rho_\varepsilon(\sigma)(x)\right| dx\right) d\sigma \leqslant \text{vol}(\Omega) \int_0^t \left[C_1 e^{-(\alpha^*/\varepsilon)\sigma} \|\rho_0\|_\infty + C_2(n_0)\varepsilon\right] d\sigma$$
$$\leqslant C_1^*(n_0)\varepsilon(1+t) \tag{20}$$

for some constant $C_1^*(n_0) > 0$, depending on the initial value $\|\rho_0\|_\infty$.

Therefore:

$$\left|y_\varepsilon(t)\right| \leqslant H_0(n_0) \int_0^t \left|y_\varepsilon(\sigma)\right| d\sigma + C_1^*(n_0)\varepsilon(1+t).$$

The Gronwall inequality provides, for $t > 0$:

$$\left|y_\varepsilon(t)\right| \leqslant C_1^*(n_0)\varepsilon(1+t)e^{H_0(n_0)t}.$$

Summarising, we have obtained the approximation result between the nonnegative solutions of the global perturbed problem and the aggregated model, as is established in the following theorem:

Theorem 5. *For each nonnegative initial data $n_0 \in C(\overline{\Omega})$, the two-time scales reaction–diffusion model* (10), (11), (12) *has a unique classical nonnegative global solution $n_\varepsilon(x,t)$ which can be written as*

$$\forall x \in \Omega,\ \forall t > 0, \quad n_\varepsilon(x,t) = \frac{1}{\text{vol}(\Omega)} N(t) + r_\varepsilon(x,t)$$

where $N(t)$ is the solution to the aggregated model (14) *corresponding to the initial data $N(0) = \int_\Omega n_0(x)\, dx$ and*

$$\sup_{x\in\Omega}\left|r_\varepsilon(x,t)\right| \leqslant a_1^*\varepsilon e^{a_2^* t} + a_3^* e^{-(\alpha^*/\varepsilon)t}, \quad t > 0,\ \varepsilon > 0$$

where a_i^*, $i = 1, 2, 3$, are positive constants depending on the initial value n_0.

Notice that this approximation result means that $n_\varepsilon(x,t)$ tends when $\varepsilon \to 0_+$ and $t > 0$ fixed, to a homogeneous spatial distribution given by the solution to the aggregated model. Moreover, this convergence is uniform with respect to x in $\overline{\Omega}$ and with respect to t on each compact interval $[t_0, T]$ with $0 < t_0 < T < +\infty$.

The particular case where the reaction term does not depend on the space variable x corresponds with the situation $\mathcal{F}(\ker A) \subset \ker A$ and recovers the formulation given in [7] and [11] for reaction–diffusion equations with large diffusivity. Roughly speaking, these authors show that the solutions to a semilinear parabolic system including a big enough diffusion term can be approximated by the solutions to an O.D.E. determined by the reaction term, which coincides with our aggregated model. A more general situation can be found in [13], where the dynamics of a class of reaction–diffusion models with large diffusivity is described by a so-called *shadow system*, whose underlying ideas are close to the construction of an aggregated model.

4. Slow–fast population models with discrete spatial structure

The aim of this section is to illustrate the fact that the abstract setting described in Section 2 also includes simpler situations in which the state space is finite-dimensional. In this case, the operator A is a matrix whose spectrum $\sigma(A)$ must satisfy some conditions that assure the essential point in our development, namely decomposition (3) of the state space in invariant *conservative* and *stable* parts. Despite the fact that this situation can be studied directly using tools from classical analysis, it is interesting from the point of view of modelling in population dynamics, as it is a suitable formulation to represent discrete spatial structure (see [5] and references therein).

To be precise, let us consider q populations ($q \geqslant 1$) living in a region divided into discrete spatial patches. The evolution processes are described by an ordinary differential system taking into account nonlinear local interactions on each patch that occur at a slow time scale and linear migration terms describing patch changes that are assumed to occur at a fast time scale. The model we are considering reads as

$$X_\varepsilon'(t) = \frac{1}{\varepsilon} A X_\varepsilon(t) + f\big(X_\varepsilon(t)\big)$$

with

$$X_\varepsilon(t) := \big(\mathbf{x}_{1\varepsilon}(t), \ldots, \mathbf{x}_{q\varepsilon}(t)\big)^T; \quad \mathbf{x}_{j\varepsilon}(t) := \big(x_{j\varepsilon}^1(t), \ldots, x_{j\varepsilon}^{N_j}(t)\big)^T$$

and where $x_{j\varepsilon}^i(t)$ is the number of individuals of population j living in the spatial patch i at time t, with $j = 1, \ldots, q$, and $N = N_1 + \cdots + N_q$ is the total number of spatial patches.

We also assume that $f : \mathbf{R}^N \longrightarrow \mathbf{R}^N$ is a locally Lipschitz continuous function and that matrix A is a block-diagonal matrix $A := \operatorname{diag}(A_1, \ldots, A_q)$ in which each diagonal block A_j has dimensions $N_j \times N_j$, and satisfies the following hypothesis:

Hypothesis 7. For each $j = 1, \ldots, q$, the following hold:

i) $\sigma(A_j) = \{0\} \cup \Lambda_j$ with $\Lambda_j \subset \{z \in \mathbf{C};\ \operatorname{Re} z < 0\}$.
ii) 0 is a simple eigenvalue of matrix A_j.

As a consequence, $\ker A_j$ is generated by an eigenvector of 0, which will be denoted by ν_j. The left eigenspace of matrix A_j associated to the eigenvalue 0 is generated by a vector ν_j^* and we choose both vectors verifying the *normalisation* condition $(\nu_j^*)^T \nu_j = 1$.

Remark. Hypothesis 7 holds for a matrix A if each diagonal block A_j is an irreducible matrix with nonnegative elements outside the diagonal and in addition satisfies that $\nu_j^* := \mathbf{1}_j^T := (1, \ldots, 1)^T \in \mathbf{R}^{N_j}$. In this case, A is a suitable matrix to represent conservative migrations between patches.

In order to simplify the calculations, we introduce the following matrices

$$\mathcal{U} := \operatorname{diag}\big(\big(\nu_1^*\big)^T \cdots \big(\nu_q^*\big)^T\big); \qquad \mathcal{V} := \operatorname{diag}(\nu_1 \cdots \nu_q)$$

which satisfy $\mathcal{U}A = 0$, $A\mathcal{V} = 0$ and $\mathcal{U}\mathcal{V} = I_q$, I_q being the $q \times q$ identity matrix.

The above considerations assure the existence of the decomposition (3) of the space $X := \mathbf{R}^N$ where $\ker A$ is a q-dimensional subspace generated by the columns of the matrix $\mathcal{V}$ and $S := \{\mathbf{v} \in \mathbf{R}^N;\ \mathcal{U}\mathbf{v} = \mathbf{0}\}$.

The *global variables* are defined by

$$\mathbf{s}(t) := \big(s_1(t), \ldots, s_q(t)\big)^T = \mathcal{U}X(t); \quad s_j(t) := \big(\nu_j^*\big)^T \mathbf{x}_j(t).$$

Notice that in the case $\nu_j^* = \mathbf{1}_j$, this set of variables represents the total number of individuals of each population. Finally, the aggregated model is given by

$$\mathbf{s}'(t) = \mathcal{U} f\big(\mathcal{V}\mathbf{s}(t)\big). \tag{21}$$

As we commented at the beginning of this section, the reduced model (21) can also be obtained by applying the general theory for aggregation methods in O.D.E. described in [5], in the case where the fast dynamics is linear. In this finite-dimensional setting it is straightforward to check the assumptions needed to apply Theorem 4 and therefore the approximation result between the asymptotic behaviour of solutions to both models holds. Also, a direct comparison result when $\varepsilon \to 0_+$ between the solutions similar to Theorem 5 can be established without major difficulties. The main point in this case is to assume supplementary smoothness conditions on function f so that global existence and boundedness of solutions to the perturbed and aggregated models provide suitable bounds similar to (18) for the *stable* part of the solution.

Finally, let us illustrate the method with the following example, which is a discrete-space version of the predator-prey model (FPP) in Section 3. The model consists of two populations of predators and preys living in a spatial region divided into two patches, connected by fast migrations:

$$\begin{cases} n_1'(t) = \dfrac{1}{\varepsilon}\big(k_{12} n_2(t) - k_{21} n_1(t)\big) + r_1 n_1(t)\left(1 - \dfrac{n_1(t)}{K_1}\right) - a_1 n_1(t) p_1(t), \\ n_2'(t) = \dfrac{1}{\varepsilon}\big(k_{21} n_1(t) - k_{12} n_2(t)\big) + r_2 n_2(t)\left(1 - \dfrac{n_2(t)}{K_2}\right) - a_2 n_2(t) p_2(t), \\ p_1'(t) = \dfrac{1}{\varepsilon}\big(m_{12} p_2(t) - m_{21} p_1(t)\big) - \mu_1 p_1(t) + b_1 n_1(t) p_1(t), \\ p_2'(t) = \dfrac{1}{\varepsilon}\big(m_{21} p_1(t) - m_{12} p_2(t)\big) - \mu_2 p_2(t) + b_2 n_2(t) p_2(t) \end{cases}$$

where $n_i(t)$, $p_i(t)$, represent the populations of preys and predators respectively at time t in patch i $(i = 1, 2)$, the positive constants k_{12}, k_{21} are the prey migration rates and the positive constants m_{12}, m_{21} are the predator dispersal rates.

Simple calculations show that the global variables are the total populations of preys and predators:

$$N(t) := n_1(t) + n_2(t); \qquad P(t) := p_1(t) + p_2(t)$$

and the aggregated model (21) coincides with the classical predator-prey model (AM) in Section 3, in which:

$$r^* := \frac{r_1 k_{12} + r_2 k_{21}}{k_{12} + k_{21}}; \qquad K^* := \frac{r^*}{(r_1/K_1)\tilde{k}_1^2 + (r_2/K_2)\tilde{k}_2^2};$$

$$a^* := a_1 \tilde{k}_1 \tilde{m}_1 + a_2 \tilde{k}_2 \tilde{m}_2; \qquad b^* := b_1 \tilde{k}_1 \tilde{m}_1 + b_2 \tilde{k}_2 \tilde{m}_2; \qquad \mu^* := \mu_1 \tilde{m}_1 + \mu_2 \tilde{m}_2$$

where

$$\tilde{k}_1 := \frac{k_{12}}{k_{12} + k_{21}}; \qquad \tilde{k}_2 := \frac{k_{21}}{k_{12} + k_{21}}; \qquad \tilde{m}_1 := \frac{m_{12}}{m_{12} + m_{21}}; \qquad \tilde{m}_2 := \frac{m_{21}}{m_{12} + m_{21}}.$$

Then, the same conclusion as in (FPP) holds for the asymptotic behaviour of the solutions to both models in this example.

5. Conclusions

In this work we develop a method to apply the so-called *aggregation of variables* theory to the simplification of an abstract semilinear evolution equation including two-time scales, defined on a Banach space. The aim of the work is to provide a unified approach to the treatment of a class of spatially structured population dynamics models, whose evolution processes occur at two different time scales, that are represented through a parameter $\varepsilon > 0$ small enough, giving rise to a mathematical singular perturbation problem.

Assuming suitable smoothness conditions on the operators that appear in the abstract formulation, the C_0-semigroup theory helps to simplify the perturbed model giving rise to a simplified *aggregated* model whose dynamics is close to the original one, when $\varepsilon \to 0_+$. Then, general results comparing the asymptotic behaviour of solutions, provide an analysis of complex singularly perturbed models in terms of simplified models.

We have illustrated the abstract approach with applications to some reaction–diffusion models including two-time scales, recovering as particular cases some well-known results on reaction–diffusion models with large diffusivity. Also, applications to population dynamics models including discrete spatial structure are given, with the aim of making evident that a wide class of two-time spatially structured population dynamics models admit a unified approach provided by the C_0-semigroup theory, as explained in Section 2.

The theory developed in this paper is based on the important assumption that the *fast* dynamics is linear and represented by an operator that is the infinitesimal generator of a C_0-semigroup to which the general theory of reduction of operators for isolated points of the spectrum can be applied. Generalisations of the method to nonlinear fast dynamics as well as abstract settings including retarded functional differential equations are in progress and will be published elsewhere.

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