Finite Size Lyapunov Exponents:

Background Theory

and

Direct Observations

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December 2010

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OUTLINE

1. Mathematical Background:
    - Dynamical Systems
    - Lyapunov Stability
    - Stable/Unstable Manifolds
    - …

2. FSLE from satellite derived velocities

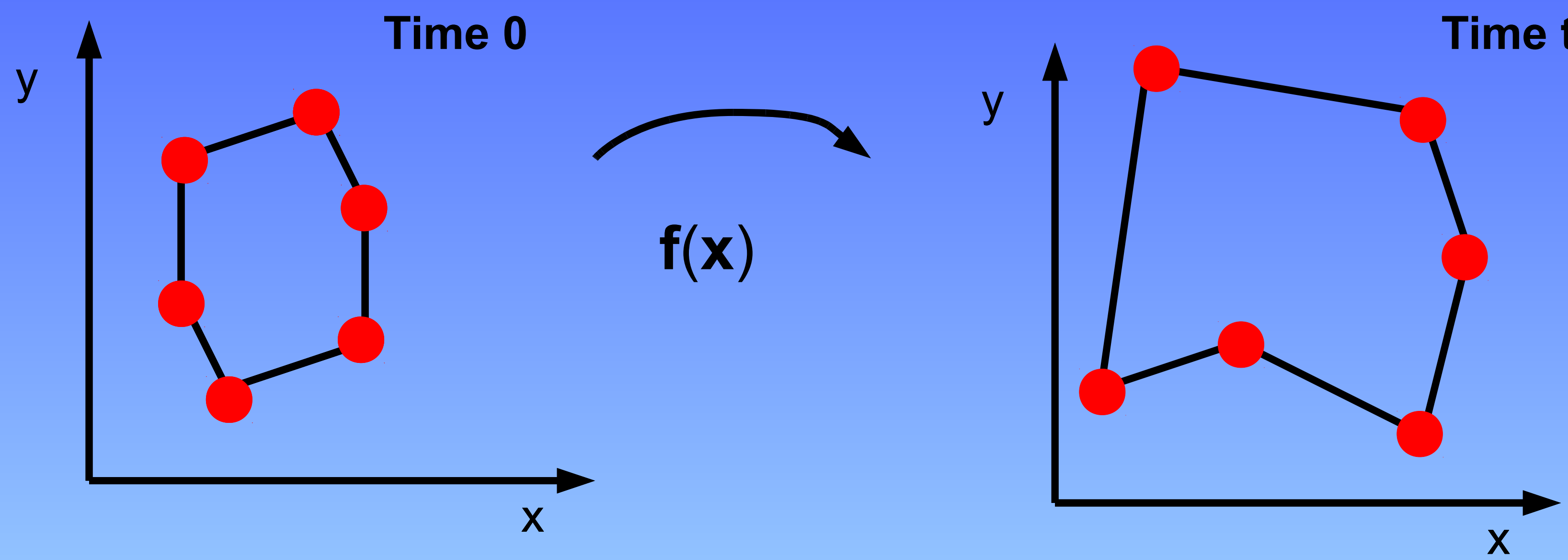
3. In-Situ Measurements (Latex10)

Basic Definitions

<u>Dynamical System</u> :

Mathematical formalization for any fixed "rule" which describes the time dependence of a point's position in its ambient space

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1 = (x_1, \dots, x_n) \\ \vdots \\ f_n = (x_1, \dots, x_n) \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$



Basic Definitions

Map/Flow :

Rule determining the evolution of the points with time

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$$

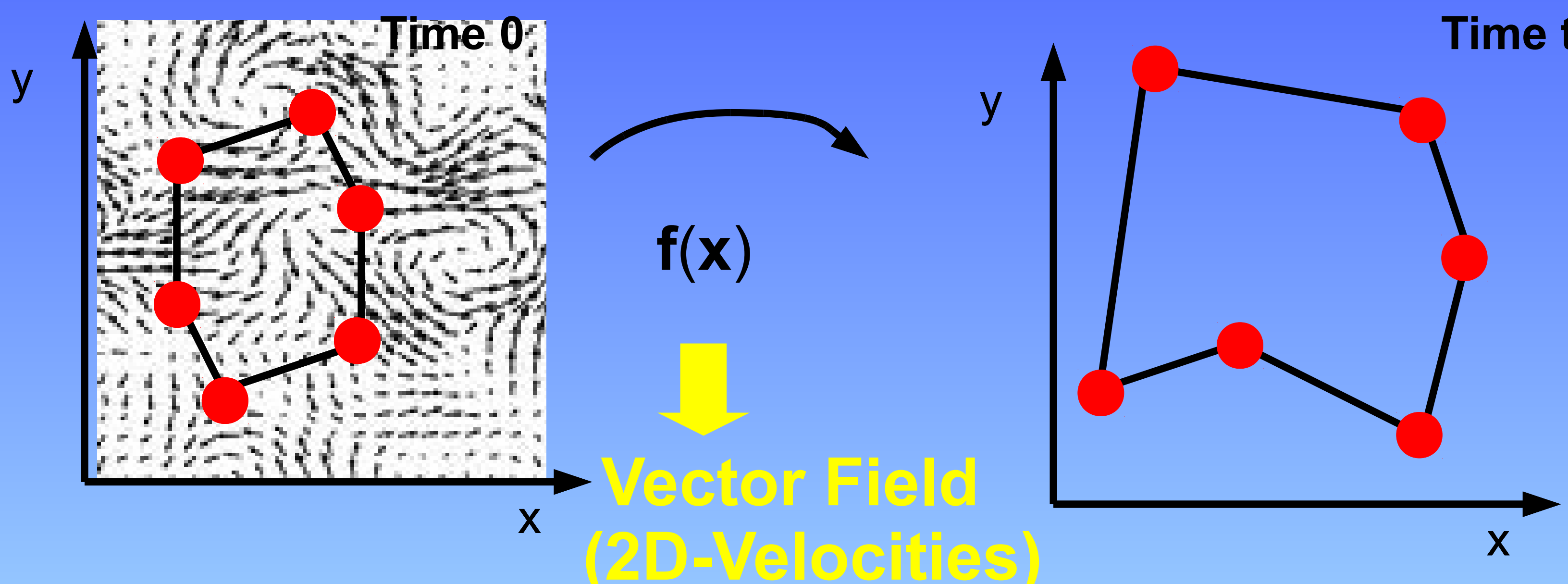
Space State :

All possible states of the system

$$\mathbf{x}(t) = \mathbf{x}(0) + \int_0^t \mathbf{f}(\mathbf{x})dt$$

Trajectory :

Temporal ordered collection of successive states



Basic Definitions

- Velocity field ➡ Turbulence ➡ Chaotic Flow

Important structures/informations on the characteristics (mixing) of the flow from stability analysis of the dynamical system around fixed points

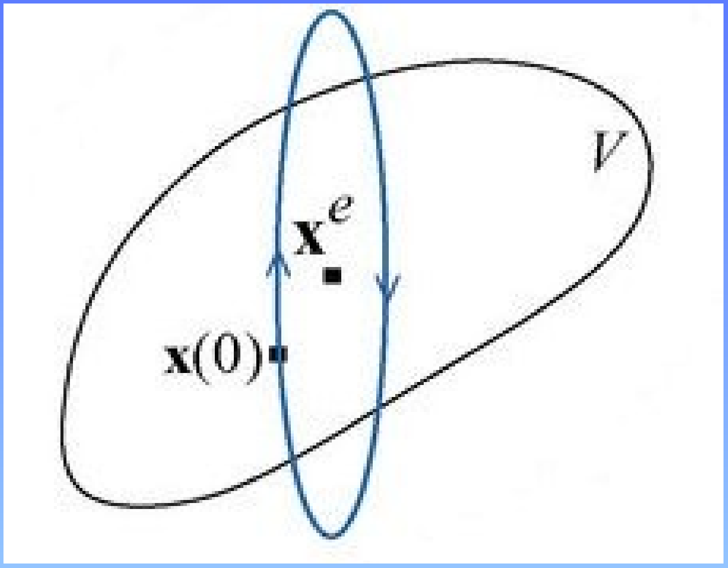
Basic Definitions

Fixed or Equilibrium point : $\mathbf{x}^e$

- Constant position in time → $\mathbf{x}(t) = \mathbf{x}^e$
- Vector field is 0 → $\dfrac{d\mathbf{x}^e}{dt} = \mathbf{f}(\mathbf{x}^e) = 0$

Lyapunov Stability :

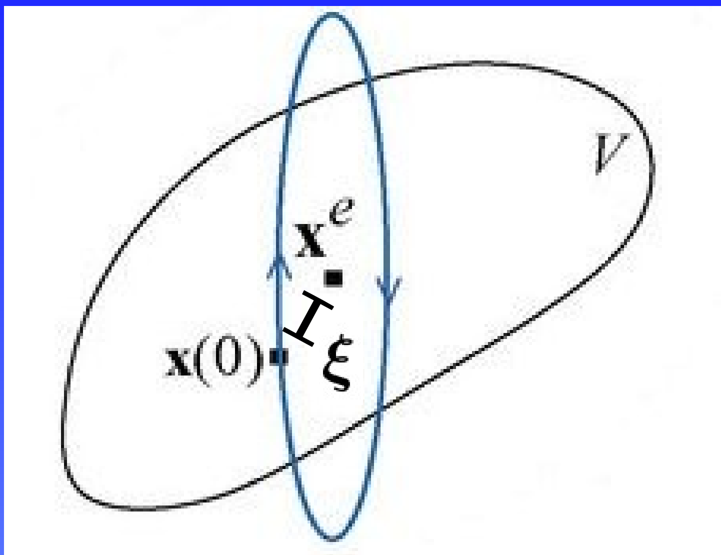
- Fixed point is a stable equilibrium point if trajectories of any point around it remain close to it with time



- Asymptotic stable
- Exponentially stable
- Unstable

Stability Analysis

Linearization :



$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$$

$$\frac{d(\mathbf{x}^e + \boldsymbol{\xi})}{dt} = \mathbf{f}(\mathbf{x}^e + \boldsymbol{\xi})$$

$$\frac{d\mathbf{x}^e}{dt} + \frac{d\boldsymbol{\xi}}{dt} = \mathbf{f}(\mathbf{x}^e) + J(\mathbf{x}^e)\boldsymbol{\xi} + \mathcal{O}(|\boldsymbol{\xi}|^2)$$

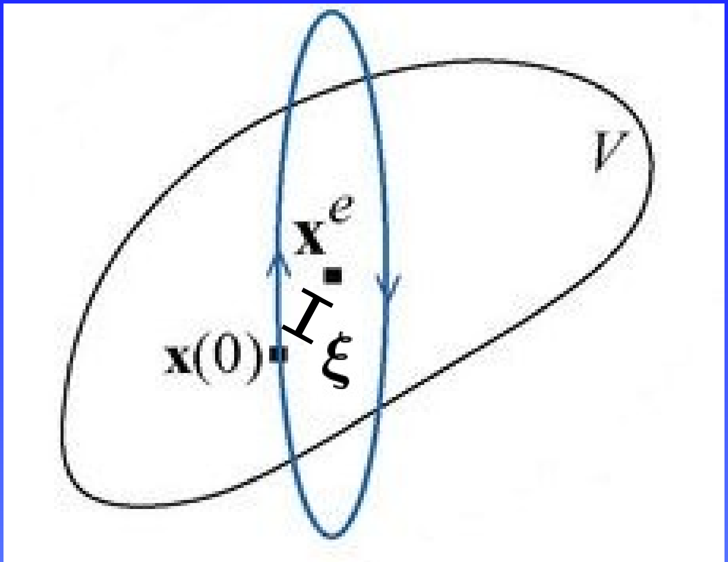
$$\frac{d\boldsymbol{\xi}}{dt} = J(\mathbf{x}^e)\boldsymbol{\xi}$$

$$J(\mathbf{x}^e) = \left(\frac{\partial f_i}{\partial x_j}\right)\bigg|_{\mathbf{x}=\mathbf{x}^e} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}\Bigg|_{\mathbf{x}=\mathbf{x}^e}$$

Jacobian matrix

Stability Analysis

Linearization :



$$\frac{d\boldsymbol{\xi}}{dt} = J(\mathbf{x}^e)\boldsymbol{\xi}$$

Solution to ODE

$$\boldsymbol{\xi}(t) = \boldsymbol{\xi}(0)\exp^{\lambda t}$$

Eigenvalues of *J* can tell if the fixed point is stable or not

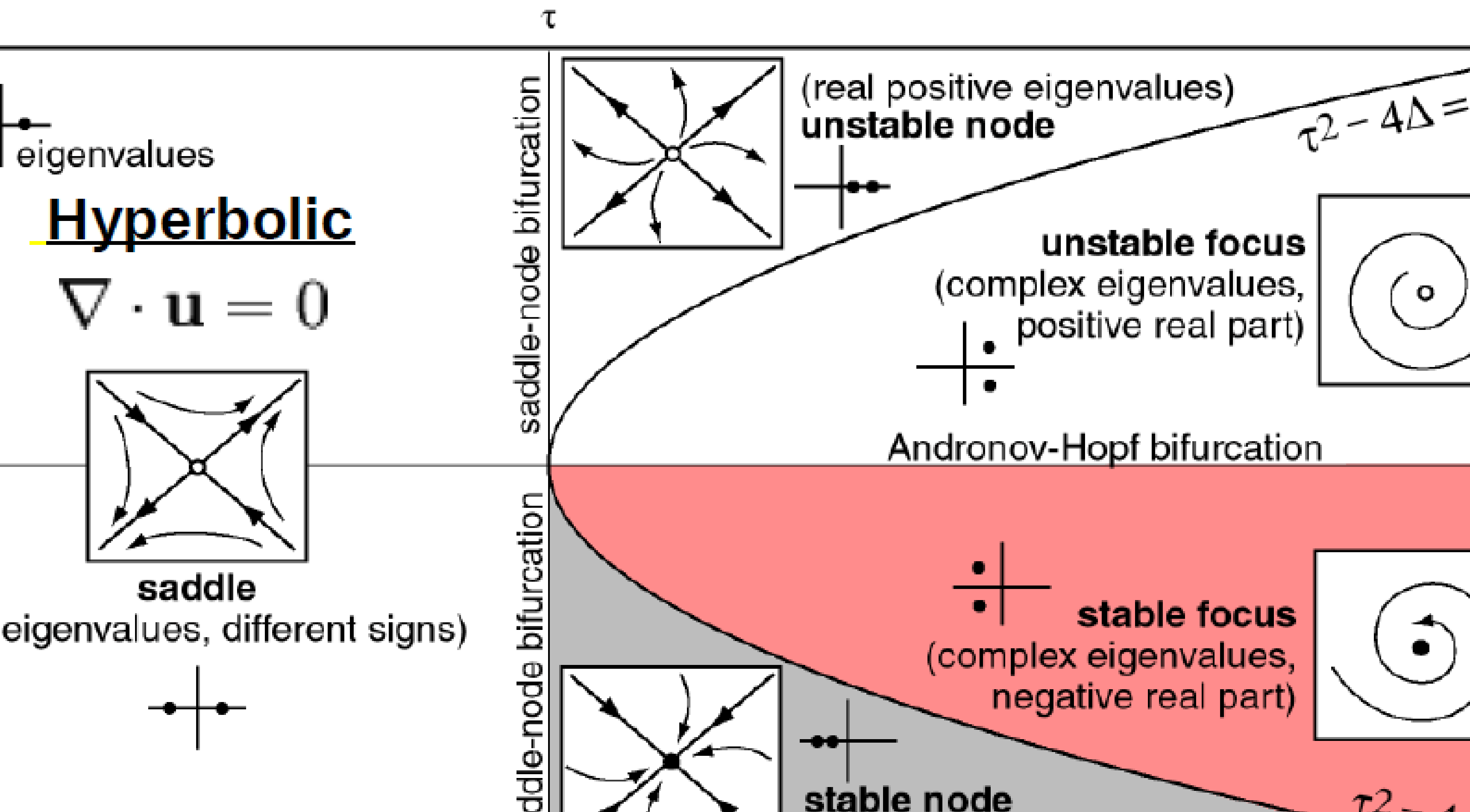
$$\det[J(\mathbf{x}^e) - \lambda I] = 0$$

- Real or imaginary
- Positive or negative

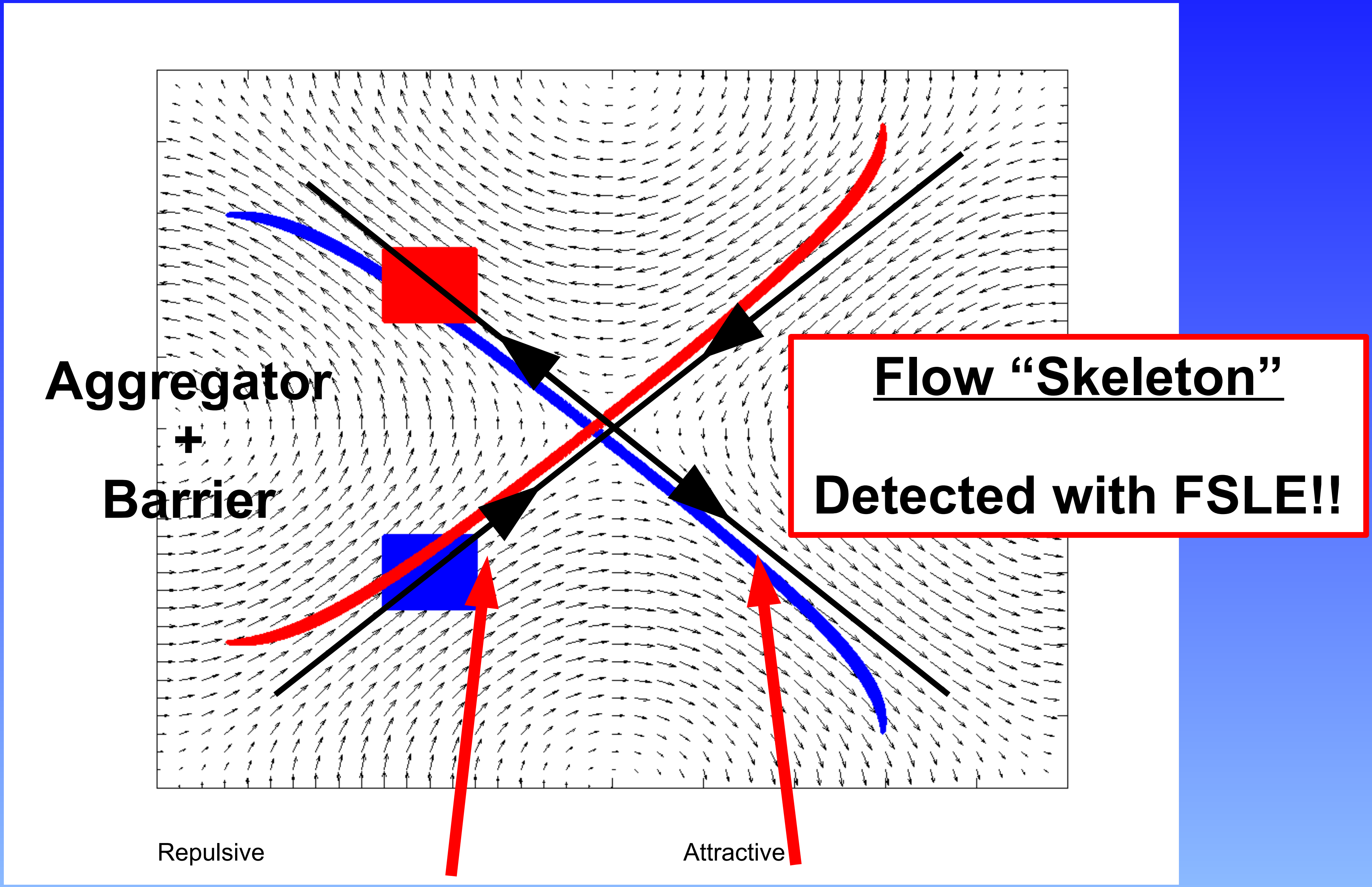
Stability Analysis

Hyperbolic

$$\nabla \cdot \mathbf{u} = 0$$



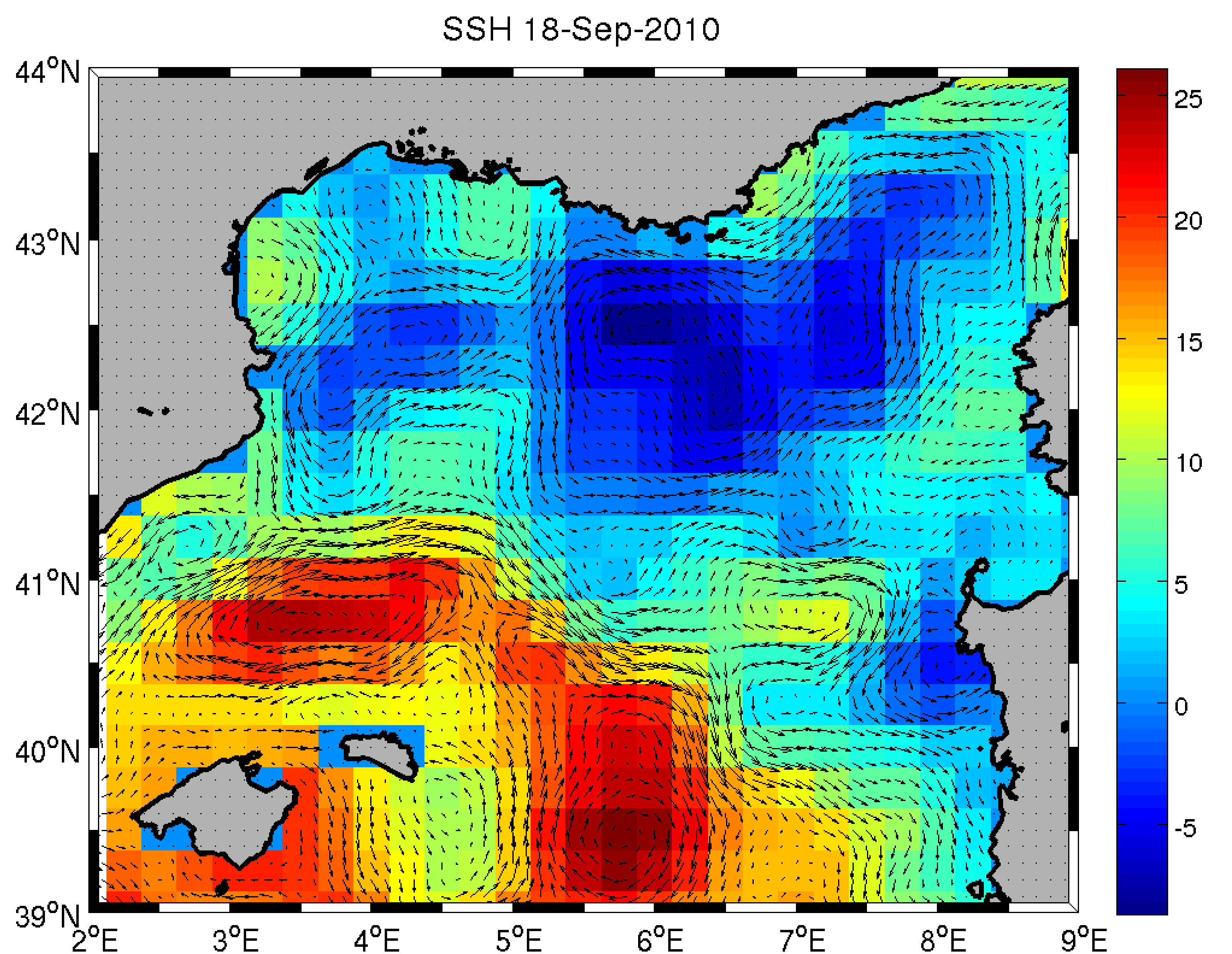
Hyperbolic Points



Stable Manifold   **Unstable Manifold**

2. FSLE

Finite Size Lyapunov Exponents

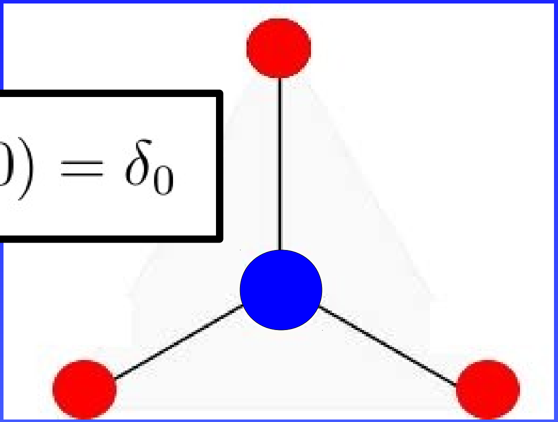


AVISO SSH (1/4 Deg)

Surface Geostrophic Velocities (1/8 Deg)

2. FSLE

Finite Size Lyapunov Exponents



$$\boldsymbol{\xi}(0) = \delta_0$$

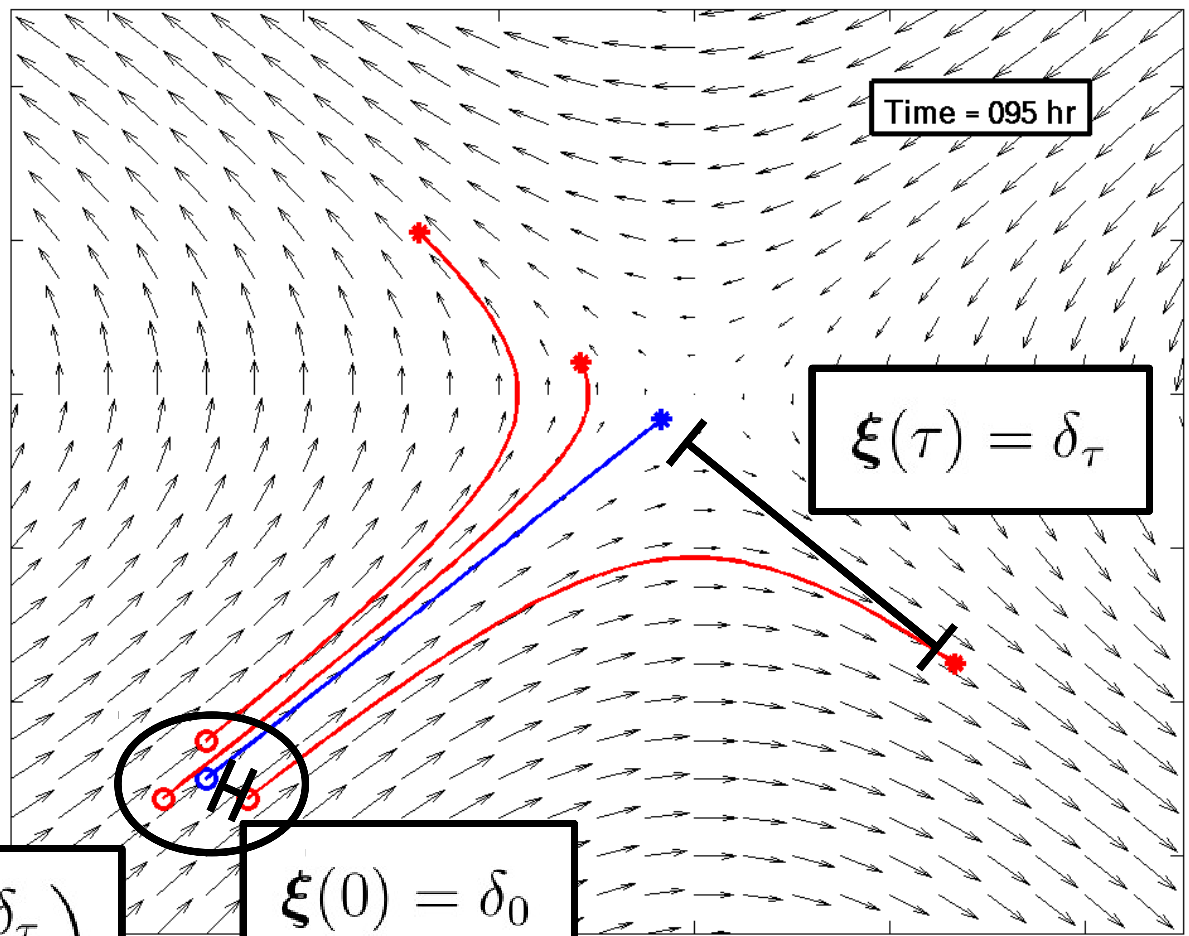
- At each grid point deployed an array of four floats
- Advected in time (forward or backward) with a Runge-Kutta 4th order (linear spatial and temporal interpolation)
- Recorded the time (tau) at which one of the distances becomes larger than a fixed spatial threshold (fixed size)

$$\boldsymbol{\xi}(\tau) = \delta_\tau$$

- Lyapunov exponent is the inverse of that time

$$\boldsymbol{\xi}(t) = \boldsymbol{\xi}(0) \exp^{\lambda t} \rightarrow \delta_\tau = \delta_0 \exp^{\lambda \tau} \rightarrow \lambda = \frac{1}{\tau} \log\left(\frac{\delta_\tau}{\delta_0}\right)$$

Finite Size Lyapunov Exponents



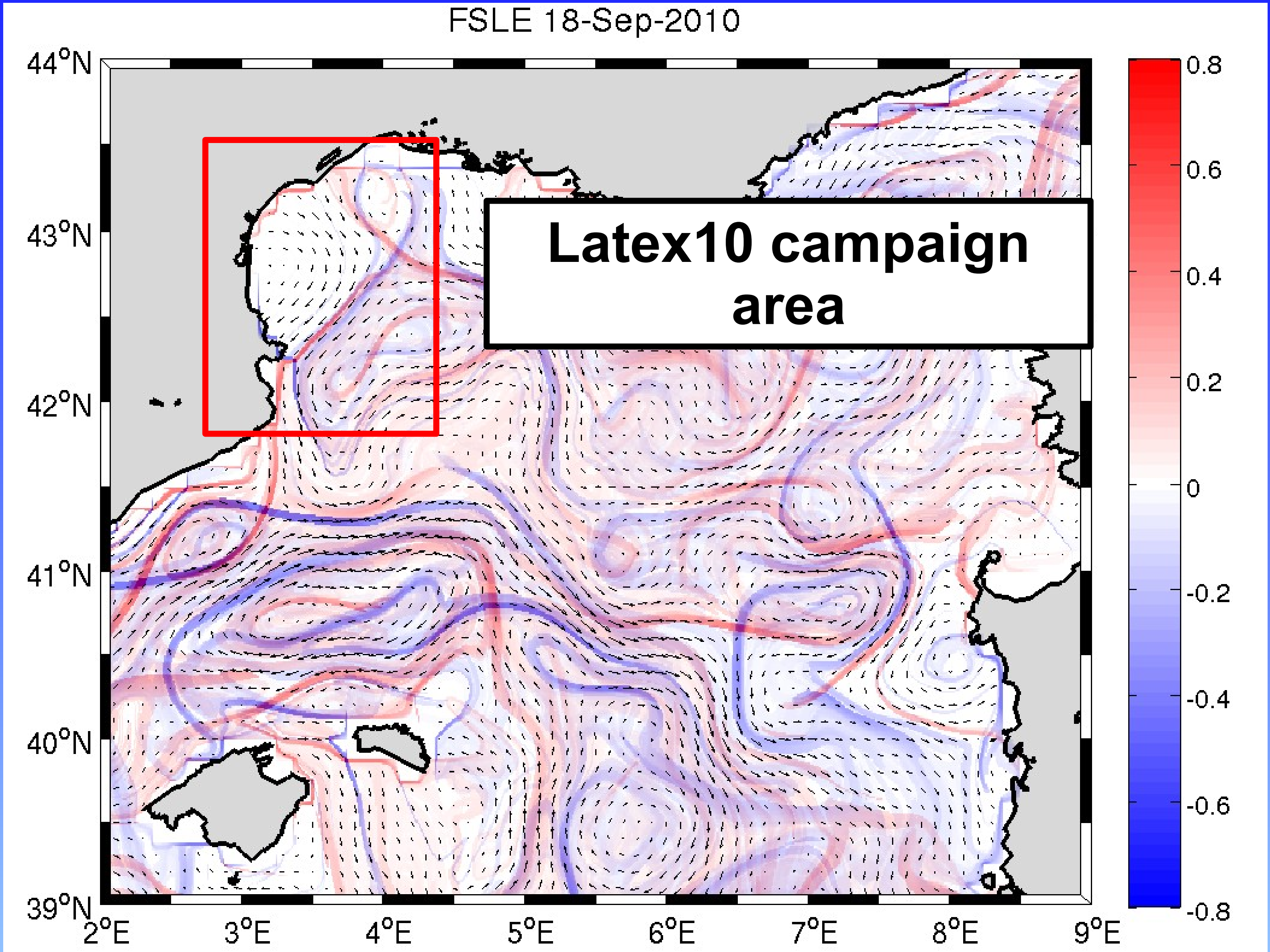
$$\boldsymbol{\xi}(\tau) = \delta_\tau$$

$$\boldsymbol{\xi}(0) = \delta_0$$

$$\lambda = \frac{1}{\tau} \log \left(\frac{\delta_\tau}{\delta_0} \right)$$

2. FSLE

Finite Size Lyapunov Exponents



Blue:
- Unstable
- Backward
(from saddle)

Red:
- Stable
- Forward
(to saddle)

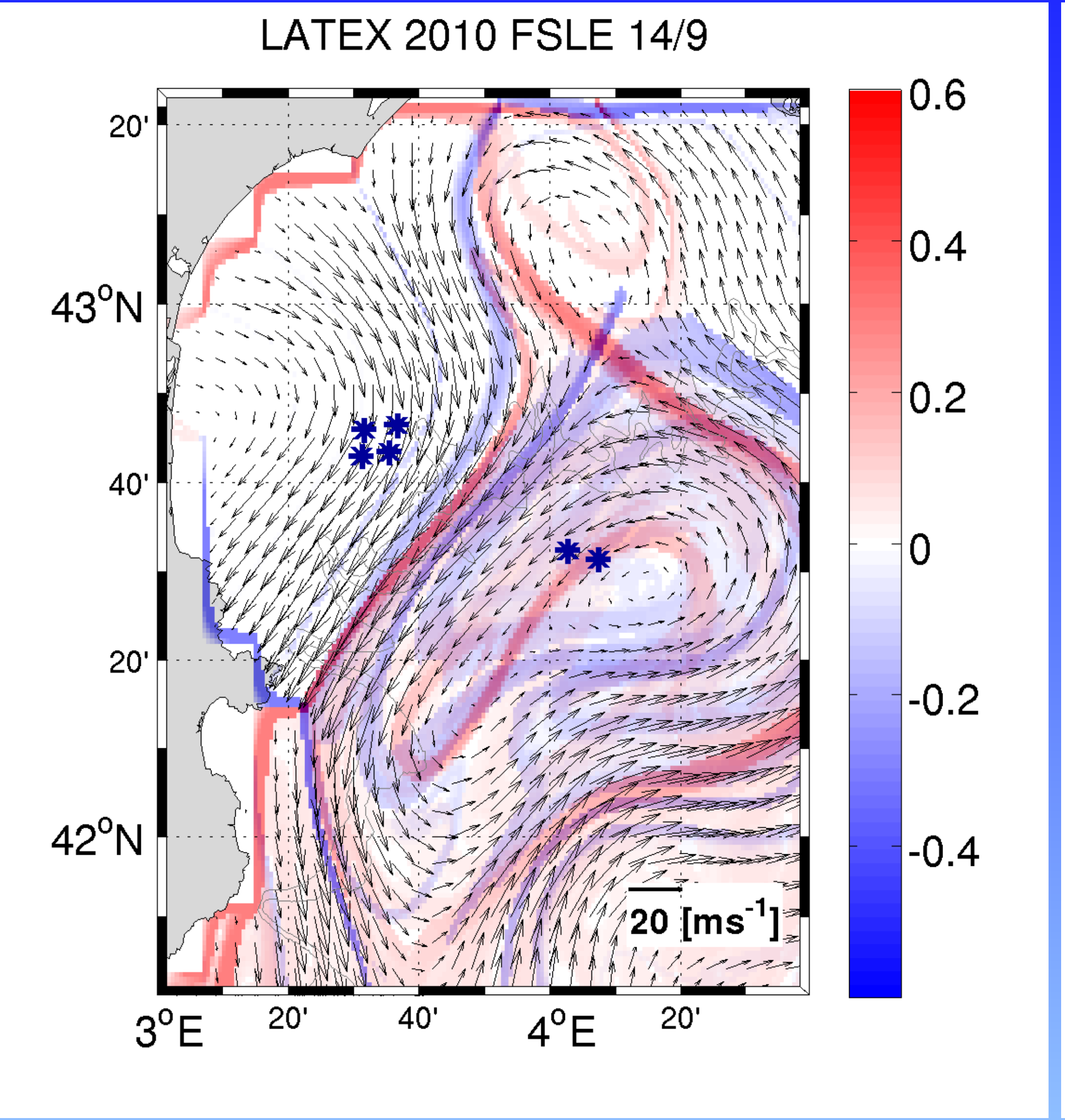
Latex10 Campaign

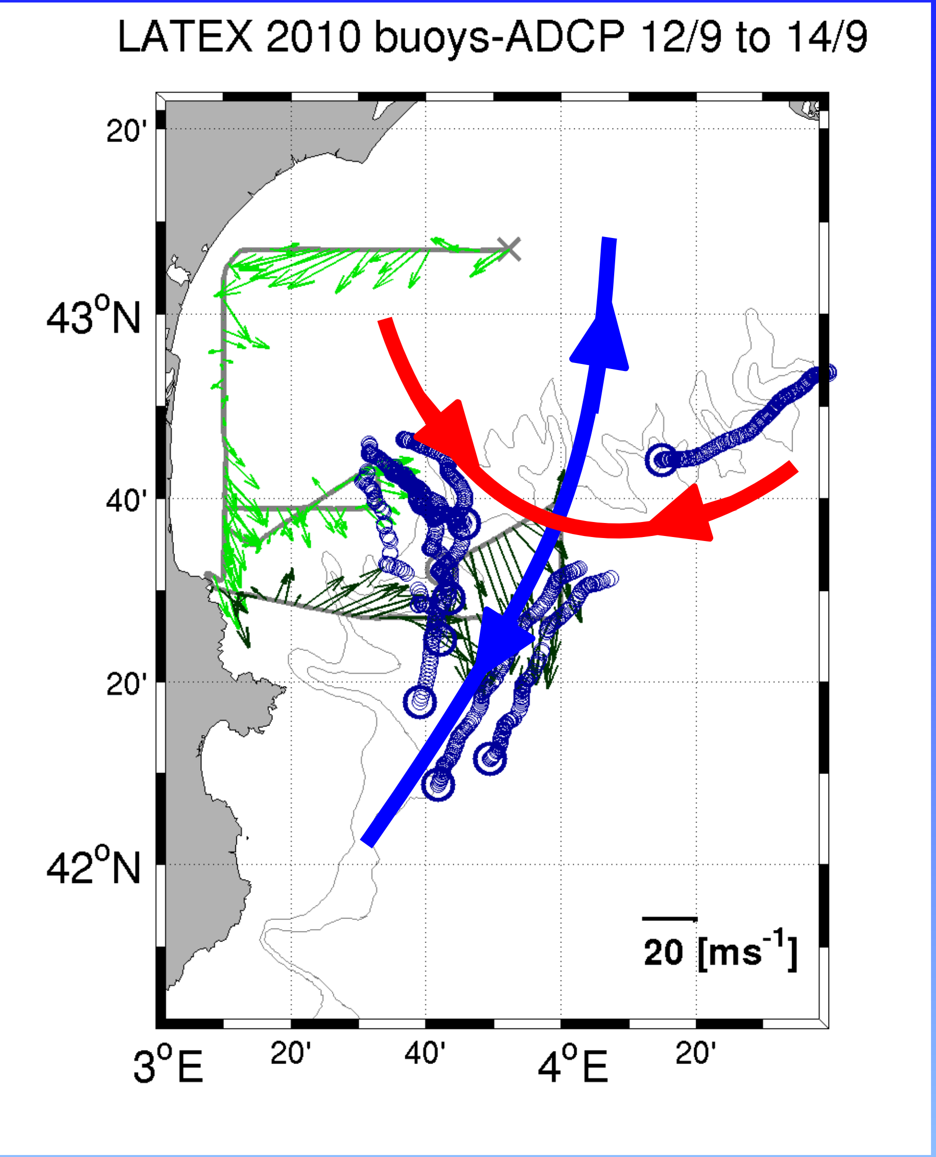
September 2010 "Lyap" experiments:

Direct observations of the manifolds computed from satellite velocities:

- **Lagrangian drifters (15 m; ARGO GPS)**
- **ADCP velocities (Real time)**

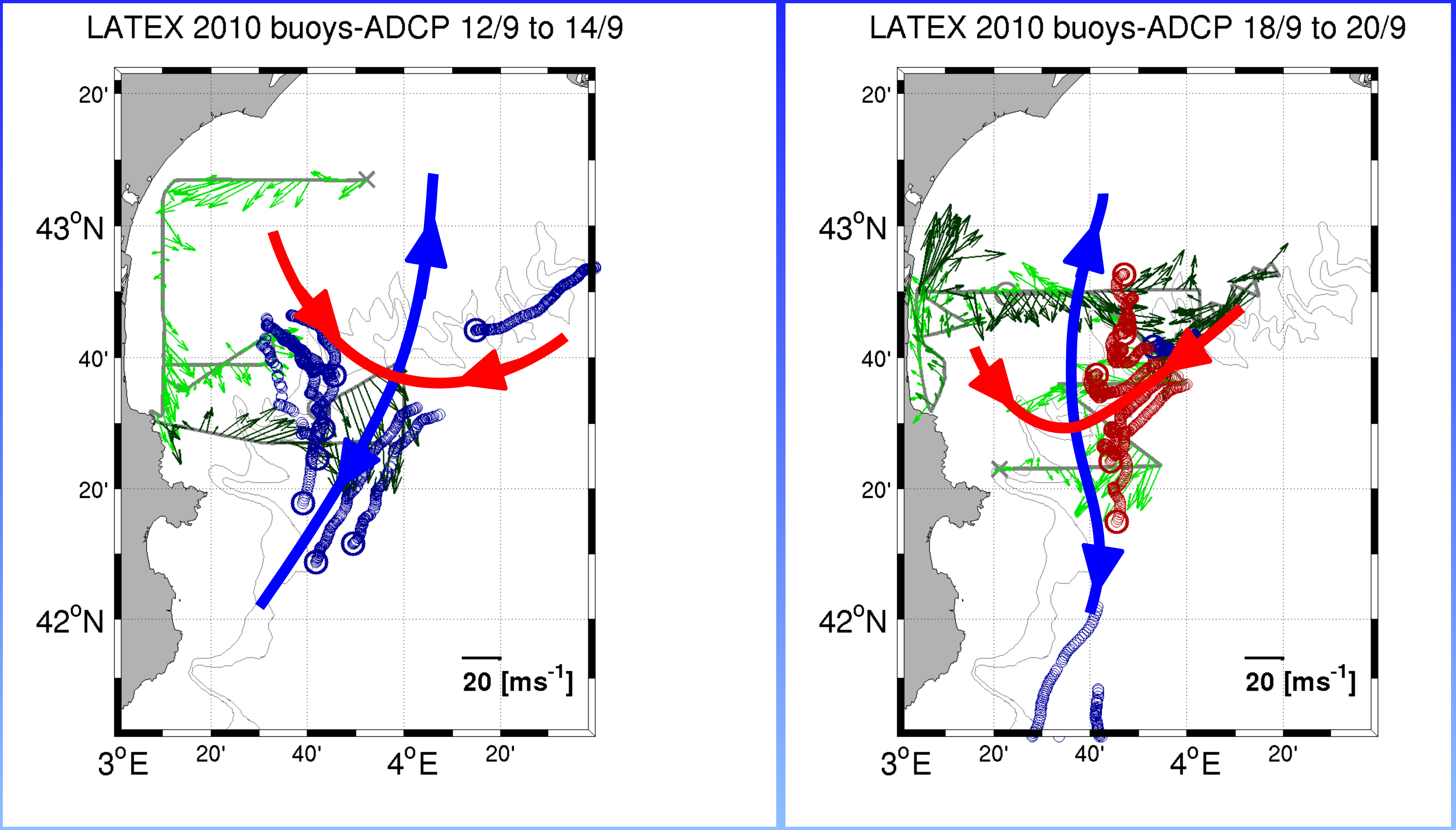
Lyap01 in LATEX



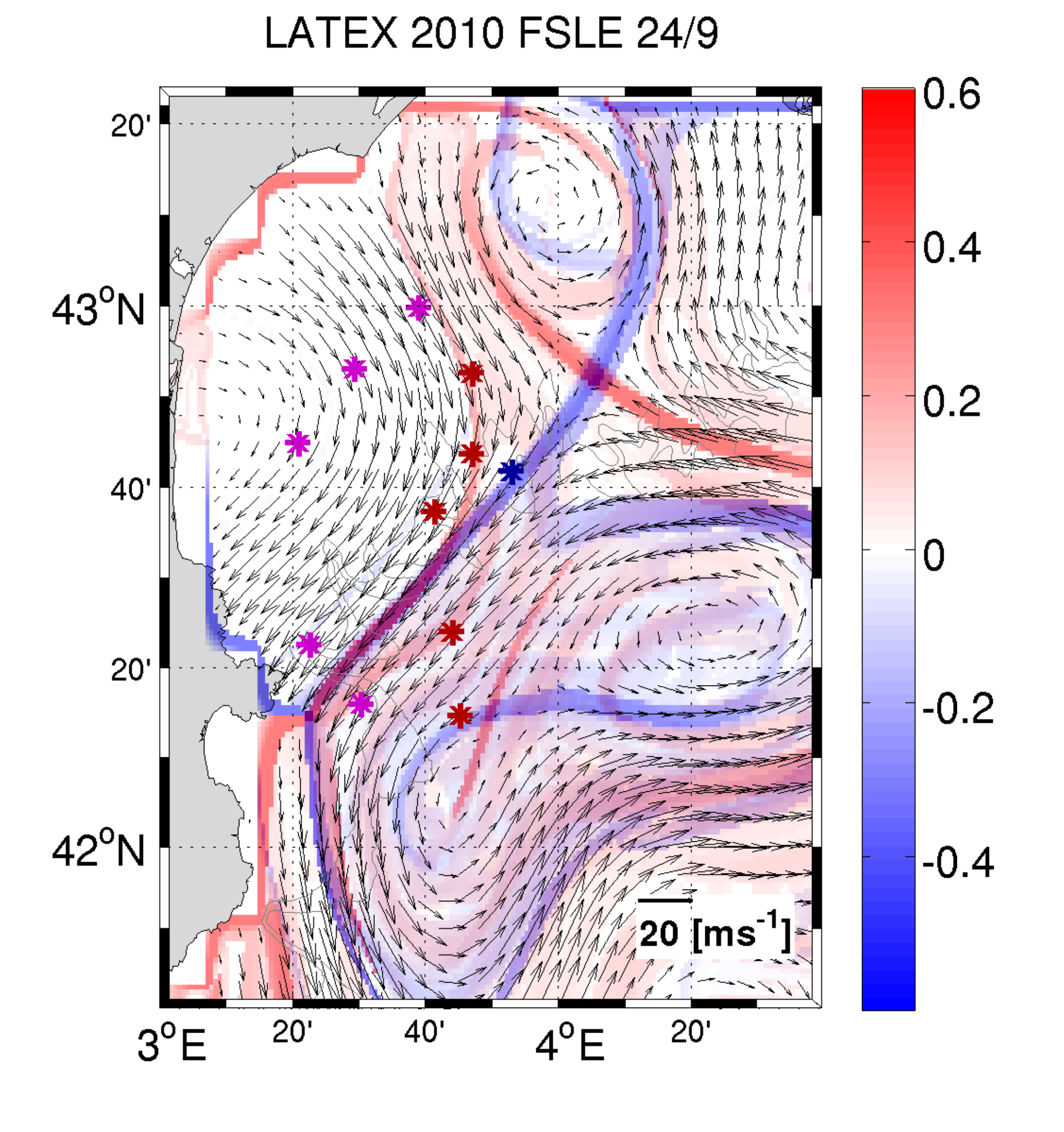


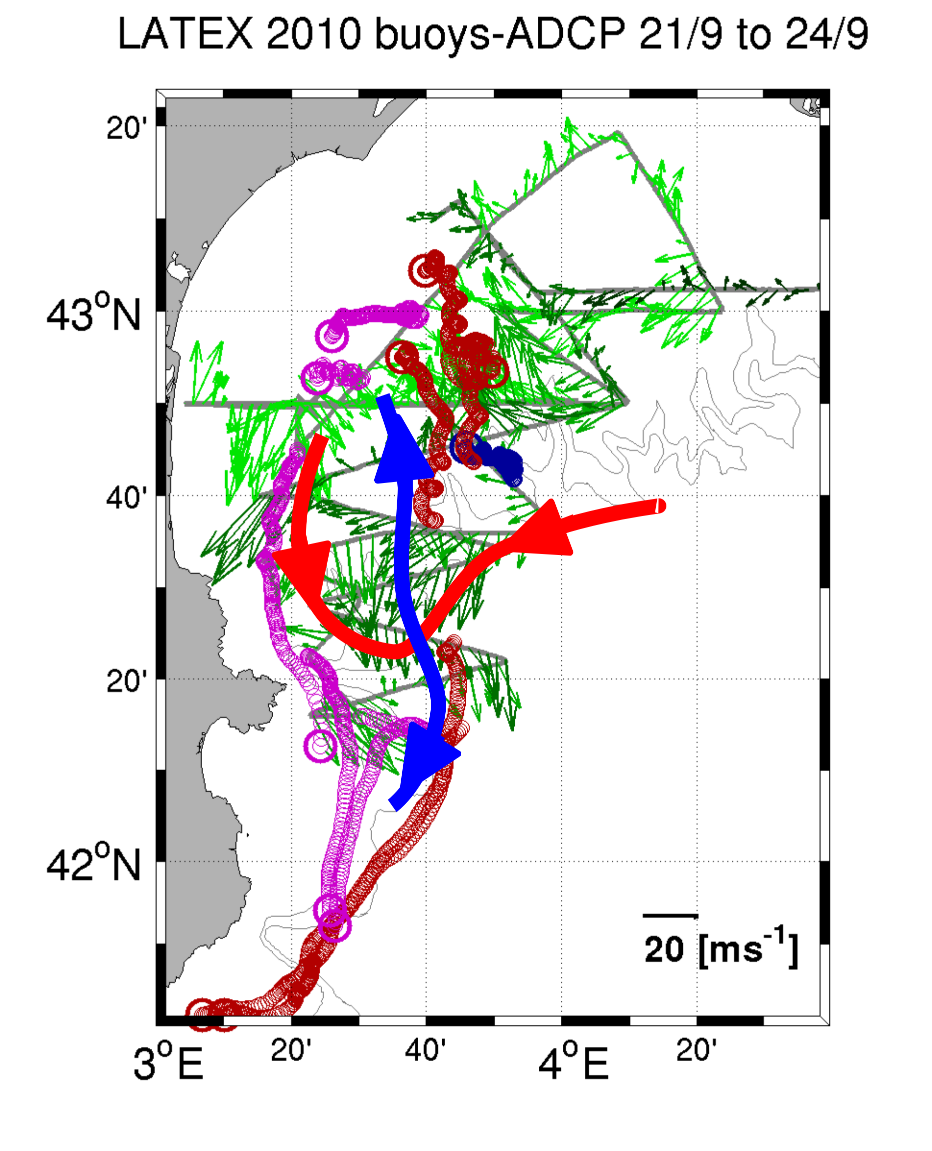
3. LATEX

Lyap02 in LATEX

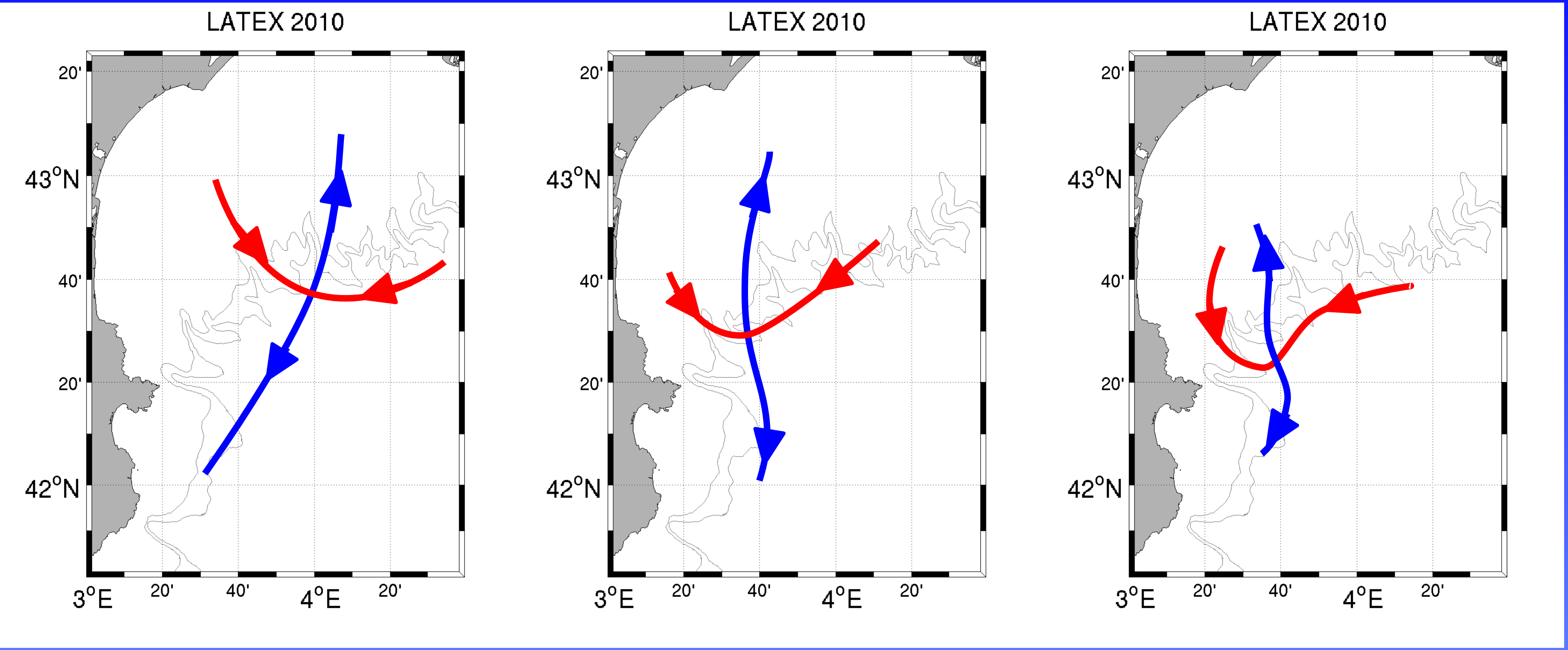


Lyap03 in LATEX



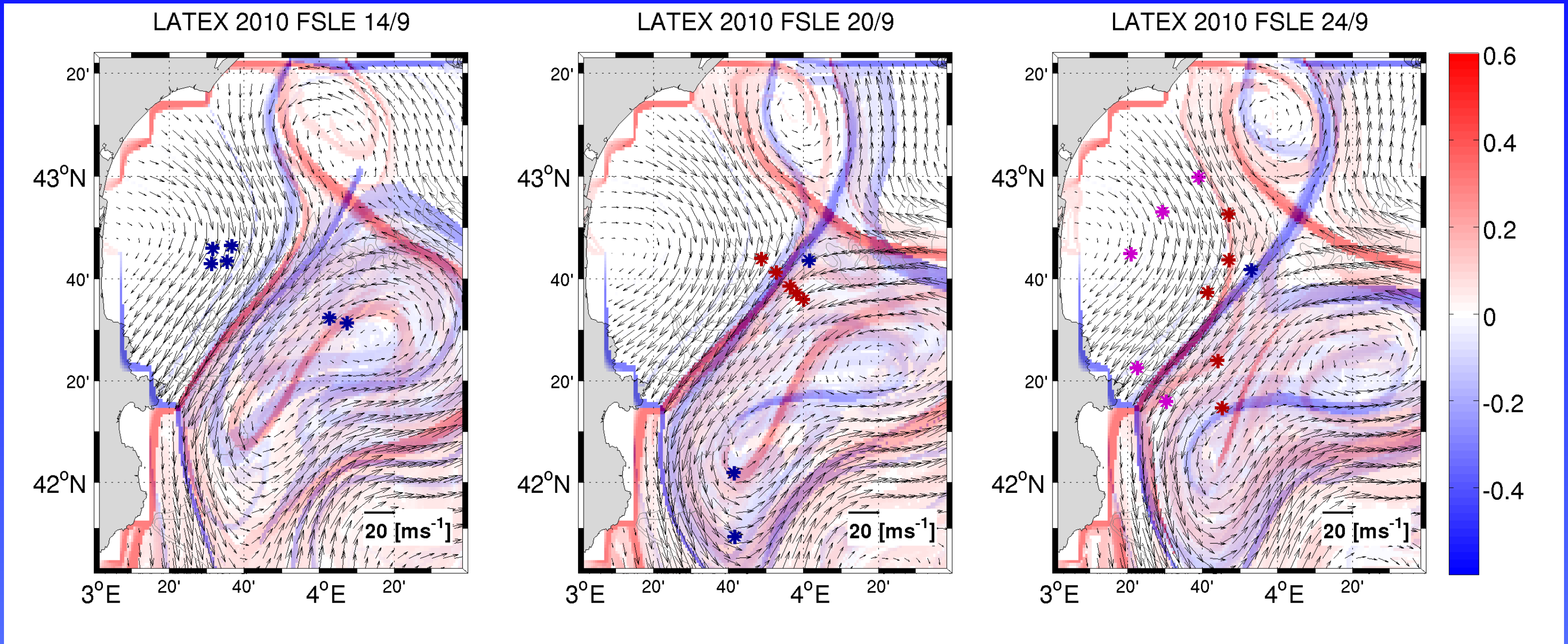


Conclusions



- **Persistent manifolds on continental shelf**
- **Hyperbolic point migration**
  (1/3 Deg in 6 days => ~ 5 cm/sec)
- **Detectable with drifters + ADCP**

Conclusions



<u>Satellite velocities</u>:

- Good for large scale
- Not so good in the coastal region
- Largest limitation for the method

Future Work

- **Improve satellite velocity field:**
  - Different processing schemes for raw data
  - Add ageostrophic components (Ekman, NIO...)
  - HF Radar velocities??
- **Numerical models:**
  - Test corrections
  - Verify hypotheses
  - Forecast models??
- **Analysis of previous Latex datasets**
- **Further Lyap experiments???**

Final goal

Method for estimate and predict transport/exchanges (pollutants, oil spill, larval transport, fisheries)

Alexandr Lyapunov

(June 6 1857 – November 3 1918)

FIN